

# Galois theories of $q$ -difference equations: comparison theorems

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## Abstract

We establish some comparison results among the different parameterized Galois theories for  $q$ -difference equations, completing the work [CHS08], that addresses the problem in the case without parameters. Our main result is the link between the abstract parameterized Galois theories, that give information on the differential properties of abstract solutions of  $q$ -difference equations, and the properties of meromorphic solutions of such equations. Notice that a linear  $q$ -difference equation with meromorphic coefficients always admits a basis of meromorphic solutions, as proven in [Pra86].

## Introduction

A linear  $q$ -difference system is a linear functional equation of the form  $\vec{y}(qx) = A(x)\vec{y}(x)$ , where, to fix ideas, we assume that  $A(x) \in \mathrm{GL}_\nu(\mathbb{C}(x))$  and that  $q \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$  is not a root of unity. This kind of functional equations appears in the literature for many reasons: they are discretization of linear differential equations that can be recovered letting  $q \rightarrow 1$  in  $\frac{\vec{y}(qx) - \vec{y}(x)}{(q-1)x}$ ; they appear in combinatorial problems in relation with  $q$ -series; they have a geometric interpretation as functional equations on the torus  $\mathbb{C}^*/q^{\mathbb{Z}}$ , when  $|q| \neq 1$ .

A functional equation is useful as far as it allows to grasp properties of its solutions. To achieve this purpose, there are two different schools: some may try to actually find explicit solutions to the equation, while others concentrate on its structural properties, that could give information on solutions that we are unable or unwilling (but most of the time unable) to write. Galois theory of difference equations follows the second line of thoughts.

Difference Galois theory has been introduced in [Fra63]. Since the first systematic work [vdPS97], which follows quite an abstract point of view, the Galois theory of  $q$ -difference equation has been developing in many directions. Theories and theorems can be classified according to different criteria, for instance:

1. After [Pra86, Theorem 3] any  $q$ -difference equations with meromorphic coefficients has a basis of meromorphic solutions. Therefore one can decide to use it to define a Galois theory, as in [Sau04], or one can define abstract solutions as in [vdPS97].
2. One can define the Galois group of  $\vec{y}(qx) = A(x)\vec{y}(x)$  as a group of automorphisms of a convenient extension containing a full set of solutions or use a more abstract Tannakian point of view.

Following these different points of view, a certain number of Galois groups have been defined in the literature. In [CHS08], the problem of establishing the isomorphisms among them is addressed. See Theorem 4.1 below.

In [HS08], the authors develop a parameterized Galois theory that takes into account the action of a derivation on a set of solutions of a  $q$ -difference equations. One of the drawbacks of this theory is that it is based on an abstract set of solutions and their derivatives, constructed in an algebra over

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a differentially closed extension of  $\mathbb{C}$ , therefore quite a huge field of definition. A natural question is to compare these abstract solutions with the Galois theory constructed using a basis of meromorphic solutions of the system.

To complete the picture, one can attach to any  $q$ -difference system several groups thanks to the theory of differential Tannakian categories introduced by Kamensky [Kam10] and Ovchinnikov in [Ovc09]. A differential Tannakian category is a generalization of the notion of Tannakian category introduced by Deligne [Del90]. As a Tannakian category (with a fiber functor) is isomorphic to the category of representations of a linear algebraic groups, a differential Tannakian category (with a differential fiber functor) is isomorphic to the category of representations of a linear differential algebraic group (see Appendix B). Any full set of solutions determines a neutral differential fiber functor on the category generated by the associated  $q$ -difference module, and hence a group. Moreover, the differential Tannakian setting allows to introduce the so called intrinsic Galois groups, associated to non-neutral fiber functors.

The main point of this paper is to establish the isomorphisms between all these groups in the literature, which should allow for a more fluid application of  $q$ -difference Galois theory, being able to exploit the different advantages of each point of view. In [PN11], the author compared some parameterized Galois groups defined over some abstract base fields and obtained the analogue of Theorem 4.3 in her setting. In this paper, we are interested in clarifying the relation between the meromorphic solutions and the abstract Galois theory, so that we can not rely on [PN11].

The paper is organized as follows. First we define the different kind of solutions and the algebras generated by them. This allows for a definition of the Galois group as a group of automorphisms. Then we recall the (differential) Tannakian formalism and the different groups that such theory allow to associate to a  $q$ -difference system. We conclude comparing all the different groups defined in the paper.

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# 1 $q$ -difference systems and their solutions

Let  $K$  be a field of characteristic zero and  $q \neq 0, 1$  be a fixed element of  $K$ . The field  $K(x)$  is naturally a  $q$ -difference field, i.e., it is equipped with the  $q$ -difference operator

$$\begin{aligned} \sigma_q : K(x) &\longrightarrow K(x), \\ f(x) &\longmapsto f(qx). \end{aligned}$$

More generally, we will call  $q$ -difference field one of the following pairs:

1. a field extension  $\mathcal{F}/K(x)$ , with a field automorphism extending the action of  $\sigma_q$ , which we will also call  $q$ -difference operator and denote by  $\sigma_q$ .
2. any sub-field stable by  $\sigma_q$  and  $\sigma_q^{-1}$  of the fields considered at the previous item, with the restriction of  $\sigma_q$ .

We denote by  $\mathcal{C}$  (or sometimes  $\mathcal{F}^{\sigma_q}$ ) the field of  $\sigma_q$ -constants of  $\mathcal{F}$ , i.e., the subfield of the elements of  $\mathcal{F}$  fixed by  $\sigma_q$ . The previous definition implies that  $\mathcal{C}$  with the identity automorphism is a  $q$ -difference field, that we will call trivial.

**Example 1.1.** Typical examples of  $q$ -difference extensions of  $K(x)$  are:

1. the field of formal Laurent series  $K((x))$ , equipped with the automorphism  $\sigma_q(\sum_n a_n x^n) = \sum_n a_n q^n x^n$ ;
2. the field  $K(x^{1/r})$ , for  $r \in \mathbb{Z}_{>1}$ , equipped with  $\sigma_q(x^{1/r}) = \tilde{q}x^{1/r}$ , for a chosen  $r$ -th root  $\tilde{q}$  of  $q$ ;
3. for  $K = \mathbb{C}$ , the field of meromorphic functions over  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .

A linear  $q$ -difference system (of order  $\nu$ ) is a functional equation of the form

$$(1.1) \quad \sigma_q(\vec{y}) = A\vec{y},$$

where  $A$  is an invertible square matrix of order  $\nu \geq 1$  with coefficients in a  $q$ -difference field  $(\mathcal{F}, \sigma_q)$  as above, i.e.,  $A \in \text{GL}_\nu(\mathcal{F})$ , and  $\vec{y}$  is a vector of unknowns. The solutions vectors are to be found in a  $q$ -difference extension of  $(\mathcal{F}, \sigma_q)$ . It is well known that they generate a vector space over  $\mathcal{C}$  of order at most  $\nu$ .

In the text below, we will consider the action of a derivation  $\partial$  on the solution set of (1.1). To do so, we will assume that there exists a derivation  $\partial$  of the field  $\mathcal{F}$ , commuting to the action of  $\sigma_q$ , that is  $\sigma_q \circ \partial = \partial \circ \sigma_q$ . We will call  $(\mathcal{F}, \sigma_q, \partial)$  a  $q$ -difference differential field or a  $(\sigma_q, \partial)$ -field, for short. Any  $q$ -difference fields of Example 1.1 is a  $(\sigma_q, \partial)$ -field endowed with the derivation  $\partial = x \frac{d}{dx}$ . Notice that, any  $q$ -difference field  $(\mathcal{F}, \sigma_q)$  can be turned in a  $q$ -difference differential field with the trivial derivation.

*Remark 1.2.* If  $\vec{y}$  is a solution vector of (1.1), then the commutativity of  $\sigma_q$  and  $\partial$  and the Leibnitz rule imply that:

$$\sigma_q \begin{pmatrix} \partial(\vec{y}) \\ \vec{y} \end{pmatrix} = \begin{pmatrix} A & \partial(A) \\ 0 & A \end{pmatrix} \begin{pmatrix} \partial(\vec{y}) \\ \vec{y} \end{pmatrix},$$

hence  $\begin{pmatrix} \partial(\vec{y}) \\ \vec{y} \end{pmatrix}$  is the solution vector of a  $q$ -difference system of order  $2\nu$ . This quite trivial remark is at the origin of more abstract constructions considered below.

We recall the basic convention of difference and differential algebra, that we will frequently use in what follows:

*Algebraic attributes always refer to the underlying ring, ideal or algebra and the operator prefix highlights the compatibility of the algebraic attribute with the operator prefix.*

For example:

- A  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra  $R$  is an  $\mathcal{F}$ -algebra equipped with a derivation  $\partial$  and  $\sigma_q$  that extend to  $R$  the action of  $\partial$  and  $\sigma_q$  on the  $(\sigma_q, \partial)$ -field  $\mathcal{F}$ .
- A  $\sigma_q$ -ideal (resp.  $(\sigma_q, \partial)$ -ideal) of a  $\sigma_q$ -ring (resp  $(\sigma_q, \partial)$ -ring)  $R$  is an ideal of  $R$  that is set-wise invariant by  $\sigma_q$  (resp. by  $\sigma_q$  and  $\partial$ ).

Since we will use more sophisticated notions of differential algebra, we have added a section on this topic in the appendix. Classical references for further readings are for instance [Kol73], [Coh65], [Lev08].

From now on we suppose that  $(\mathcal{F}, \sigma_q, \partial)$  is a  $q$ -difference differential field of characteristic zero and that we are given a  $q$ -difference system of the form (1.1).

## 1.1 Parameterized Picard-Vessiot rings

In this section we recall some results on the existence of abstract solutions for linear  $q$ -difference systems endowed with the action of a derivation. For a quick survey of the needed notions of differential algebra, we refer to Appendix A.

**Definition 1.3.** A  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra  $R$  is a *parameterized Picard-Vessiot ring* for (1.1) if

1.  $R$  is a simple  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra, i.e., there are no non-trivial ideal being set-wise fixed by  $\sigma_q$  and  $\partial$ ;
2. there exists a  $Z \in \mathrm{GL}_\nu(R)$  such that  $\sigma_q(Z) = AZ$ ;
3.  $R = \mathcal{F}\{Z, \det Z^{-1}\}_\partial$ , i.e.,  $R$  is generated as a  $\mathcal{F}$ -algebra by the entries of  $Z$ , the inverse of the determinant of  $Z$  and all their derivatives with respect to  $\partial$  (see Definition A.1).

Such a ring always exists: We consider the ring of differential polynomials  $S = \mathcal{F}\{Y, \det Y^{-1}\}_\partial$ , where  $Y$  is a matrix of differential indeterminates over  $\mathcal{F}$  of order  $\nu$ . A  $q$ -difference operator compatible with the differential structure of the  $\partial$ - $\mathcal{F}$ -algebra  $S$  can be defined setting:

$$\begin{aligned}\sigma_q(Y) &= AY, \\ \sigma_q(\partial Y) &= \partial(\sigma_q Y) = A\partial(Y) + \partial(A)Y, \\ &\text{and so on, using the commutativity of } \sigma_q \text{ and } \partial \text{ and the Leibnitz rule.}\end{aligned}$$

Any quotient of the ring  $S$  by a maximal  $(\sigma_q, \partial)$ -ideal, i.e., a maximal ideal in the set of  $(\sigma_q, \partial)$ -ideals, is a  $(\sigma_q, \partial)$ -Picard-Vessiot ring.

**Lemma 1.4** ([HS08, Lemma 6.8]). *Any simple  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra  $R$  that is finitely generated as  $\partial$ - $\mathcal{F}$ -algebra, has the following structure: There exist a positive integer  $t$  and  $e_0, \dots, e_{t-1}$  idempotents of  $R$ , generating the ideals  $R_i := e_i R$ , such that:*

1.  $R = R_0 \oplus \dots \oplus R_{t-1}$ ;
2.  $\sigma_q$  permutes transitively the set  $\{R_0, \dots, R_{t-1}\}$  and  $\sigma_q^t$  leaves each  $R_i$  invariant;
3. each  $R_i$  is a domain and a simple  $(\sigma_q^t, \partial)$ - $\mathcal{F}$ -algebra.

A  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra satisfying the three properties of the lemma above is called a  $(\sigma_q, \partial)$ - $\mathcal{F}$ -pseudo-domain, by analogy with the definition of  $\sigma_q$ - $\mathcal{F}$ -pseudo-domain. See [Wib12a, §1.1]. We immediately obtain:

**Corollary 1.5.** *A parameterized Picard-Vessiot ring is a  $(\sigma_q, \partial)$ - $\mathcal{F}$ -pseudo domain.*

Since  $\sigma_q$  and  $\partial$  commute, the field  $\mathcal{C}$  is naturally a  $\partial$ -field. If it is  $\partial$ -closed (see Definition A.2) we have:

**Proposition 1.6** ([HS08, Proposition 2.4]). *If  $\mathcal{C}$  is a  $\partial$ -closed field, then the  $\sigma_q$ -constants of any parameterized Picard-Vessiot ring coincide with  $\mathcal{C}$ . Moreover, any two parameterized Picard-Vessiot rings are isomorphic as  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebras.*

In analogy with [CHS08, Definition 2.1] we set:

**Definition 1.7.** A  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra  $R$  is a *weak parameterized Picard-Vessiot ring* for (1.1) if

1.  $R = \mathcal{F}\{Z, \det Z^{-1}\}_\partial$ , where  $Z \in \mathrm{GL}_\nu(R)$  and  $\sigma_q(Z) = AZ$ ;
2.  $\mathcal{C} := \mathcal{F}^{\sigma_q} = R^{\sigma_q}$ .

Proposition 1.6 says that, if we have a differentially closed field of  $\sigma_q$ -constants, a parameterized Picard-Vessiot ring is always a weak parameterized Picard-Vessiot ring. However M. Wibmer has proved that assuming that  $\mathcal{C}$  is algebraically closed is enough to ensure the existence:

**Proposition 1.8** ([Wib12b, Corollary 9] and [DVH12, Proposition 1.16 and Corollary 1.19])). *If  $\mathcal{C}$  is algebraically closed, there exists a weak parameterized Picard-Vessiot ring  $R$ , which is moreover  $\sigma_q$ -simple, i.e., has no non-trivial  $\sigma_q$ -ideals.*

*Remark 1.9.* Notice that the proposition above proves more than needed. Indeed the weak parameterized Picard-Vessiot ring constructed in Proposition 1.8 is  $\sigma_q$ -simple, hence it is *a fortiori* a simple  $(\sigma_q, \partial)$ - $\mathcal{F}$ -algebra. This means that it is also a parameterized Picard-Vessiot ring in the sense of Definition 1.3. While it is relatively easy to construct a  $(\sigma_q, \partial)$ -simple parameterized Picard-Vessiot ring, Wibmer's idea for the construction of  $\sigma_q$ -simple parameterized Picard-Vessiot ring is quite subtle.

If  $\mathcal{C}$  is algebraically closed, uniqueness is assured only after extension to a differential closure of  $\mathcal{C}$ . See Proposition 1.6 above and [Wib12b, page 164].

We can now define the parameterized difference Galois group:

**Definition 1.10.** Let  $R$  be a weak parameterized Picard-Vessiot ring for a  $q$ -difference system (1.1) defined over  $\mathcal{F}$ . We define the parameterized difference Galois group of  $R/\mathcal{F}$  as the covariant functor:

$$G_R^\partial : \begin{array}{ccc} \partial\text{-}\mathcal{C}\text{-algebras} & \rightarrow & \text{Groups} \\ \mathcal{S} & \mapsto & \text{Aut}_{\mathcal{F} \otimes \mathcal{S}}^{(\sigma_q, \partial)}(R \otimes_{\mathcal{C}} \mathcal{S}), \end{array}$$

where: the derivation  $\partial$  of  $R \otimes_{\mathcal{C}} \mathcal{S}$  is defined by  $\partial_R \otimes id + id \otimes \partial_{\mathcal{S}}$ ; the operator  $\sigma_q$  extends from  $R$  to  $R \otimes_{\mathcal{C}} \mathcal{S}$  by linearity, acting trivially over  $\mathcal{S}$ ; the notation  $\text{Aut}_{\mathcal{F} \otimes \mathcal{S}}^{(\sigma_q, \partial)}(R \otimes_{\mathcal{C}} \mathcal{S})$  stands for the group of  $(\sigma_q, \partial)$ - $\mathcal{F} \otimes_{\mathcal{C}} \mathcal{S}$ -automorphisms of  $R \otimes_{\mathcal{C}} \mathcal{S}$ .

We refer to Appendix B for the geometric definition used in the following proposition:

**Proposition 1.11.** *The parameterized difference Galois group  $G_R^\partial$  is representable by a  $\partial$ - $\mathcal{C}$ -subgroup scheme of  $\text{GL}_\nu(\mathcal{C})$ .*

*Proof.* We omit this proof which is a straightforward parameterized analog of [CHS08, Proposition 2.2].  $\square$

*Remark 1.12.* Let us suppose that  $(\mathcal{F}, \sigma_q, \partial)$  is a  $q$ -difference field with a trivial derivation  $\partial$  and let us consider the field of differential polynomials  $S = \mathcal{F}\{Y, \det Y^{-1}\}_\partial$  with the  $q$ -difference structure induced by (1.1). By construction, the derivation on  $S$  is non-trivial, indeed  $\partial Y$  is non-zero. The  $\partial$ -ideal generated by  $\partial Y$  is a  $(\sigma_q, \partial)$ -ideal, in fact we have:

$$\sigma_q(\partial Y) = \partial(A)Y + A(\partial Y) = A(\partial Y) \text{ and hence } \sigma_q(\partial^k Y) = A(\partial^k Y) \text{ for all } k \in \mathbb{N}.$$

Moreover the quotient  $S/(\partial Y)$  is nothing else than the ring of polynomials  $\mathcal{F}[Y, \det Y^{-1}]$ , endowed with the  $q$ -difference structure given by  $\sigma_q(Y) = AY$  and the trivial derivation. Any of its maximal  $\sigma_q$ -ideals is the quotient of a maximal  $(\sigma_q, \partial)$ -ideal of  $S$  by  $(\partial Y)$ . The reader familiar with the Galois theory of difference equations will have already noticed that the parameterized Picard-Vessiot ring that we have constructed in this way is actually a usual Picard-Vessiot ring for (1.1) over  $\mathcal{F}$  (see [vdPS97]). In this sense, we say that, when  $\partial$  is the trivial derivation, a (weak) parameterized Picard-Vessiot ring allows to recover a usual (weak) Picard-Vessiot ring. Then the definition above boils down to the definition of the usual difference Galois group, which is representable by a linear algebraic group.

This is true in a more general sense. For a general derivation  $\partial$  and for a given (weak) parameterized Picard-Vessiot ring  $R = \mathcal{F}\{Z, \det Z^{-1}\}_\partial$  for (1.1) over  $\mathcal{F}$ , we can consider the subalgebra  $R_0 = \mathcal{F}[Z, \det Z^{-1}]$  of  $R$ . This is the usual (weak) Picard-Vessiot ring for (1.1) over  $\mathcal{F}$  (see [HS08, Proposition 2.8]). We remind that the difference Galois group  $G_R$  of  $R_0/\mathcal{F}$  is defined as follows

$$(1.2) \quad G_R : \begin{array}{ccc} \mathcal{C}\text{-algebras} & \rightarrow & \text{Groups} \\ \mathcal{S} & \mapsto & \text{Aut}_{\mathcal{F} \otimes \mathcal{S}}^{\sigma_q}(R_0 \otimes_{\mathcal{C}} \mathcal{S}), \end{array}$$

where the operator  $\sigma_q$  extends from  $R_0$  to  $R_0 \otimes_{\mathcal{C}} \mathcal{S}$  by linearity, acting trivially over  $\mathcal{S}$ . The difference Galois group  $G_R$  is representable by a group-scheme defined over  $\mathcal{C}$  (see [CHS08, Prop. 2.2]).

Notice that  $G_R$  is an abuse of notation and we should write  $G_{R_0}$  instead. We prefer not to use a complicate notation, since there will be no confusion in the text below.

## 1.2 Weak parameterized Picard-Vessiot rings associated with meromorphic solutions

We assume that our base  $q$ -difference field  $(\mathcal{F}, \sigma_q)$  is a subfield of the field  $\mathcal{M}(\mathbb{C}^*)$  of meromorphic functions over  $\mathbb{C}^*$  and that  $|q| \neq 0, 1$ .

*Remark 1.13.* This is a restrictive assumption, but not as much as one could imagine. Later on we will focus on a  $q$ -difference field  $(K(x), \sigma_q)$  of rational functions, where  $K$  is a finitely generated extension of  $\mathbb{Q}$ . Of course we can always embed  $K$  into  $\mathbb{C}$ . If  $q$  is transcendental over  $\mathbb{Q}$ , then we can always choose an embedding such that  $q$  will have an image in  $\mathbb{C}$  of norm different than 1. Of course, if  $q$  is algebraic this is possible “most of the times” but not always. See also Remark 1.16.

We consider the elliptic curve  $E := \mathbb{C}^*/q^{\mathbb{Z}}$  and its field  $\mathbb{C}_E$  of elliptic functions, that is the meromorphic functions over  $\mathbb{C}^*$  that are invariant by  $\sigma_q$ . We recall the following result:

**Theorem 1.14** ([Pra86, Theorem 3]). *Any linear  $q$ -difference system  $\sigma_q(\vec{y}) = A\vec{y}$ , with  $A \in \mathrm{GL}_\nu(\mathcal{M}(\mathbb{C}^*))$ , has a basis of solutions with coefficients in  $\mathcal{M}(\mathbb{C}^*)$ , linearly independent over  $\mathbb{C}_E$ .*

The theorem above requires some comments. By a full basis of linearly independent solutions we mean  $\nu$  solution vectors  $\vec{y}_1, \dots, \vec{y}_\nu \in \mathcal{M}(\mathbb{C}^*)$ , linearly independent over  $\mathbb{C}_E$ . One usually say that  $\sigma_q(\vec{y}) = A\vec{y}$  admits a *fundamental matrix* of solutions  $Y \in \mathrm{GL}_\nu(\mathcal{M}(\mathbb{C}^*))$ , whose columns are  $\vec{y}_1, \dots, \vec{y}_\nu \in \mathcal{M}(\mathbb{C}^*)$ , which summarize the conclusion of the theorem. To the best of our knowledge, there is no constructive proof of the existence of a basis of meromorphic solutions of a general  $q$ -difference system with meromorphic coefficients. We are able to do it in full generality under the assumption that  $A \in \mathrm{GL}_\nu(\mathbb{C}(x))$  (see [Dre15]).

**Example 1.15.** Let us assume that  $|q| > 1$ . The Jacobi theta function  $\Theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n$  is an element of  $\mathcal{M}(\mathbb{C}^*)$ . It is solution of the  $q$ -difference equation  $y(qx) = qx y(x)$ . Following [Sau00], one can use the following meromorphic functions

- $\Theta_q(cx)/\Theta_q(x)$ , with  $c \in \mathbb{C}^*$ , solution of  $y(qx) = cy(x)$ ,
- $x\Theta'_q(x)/\Theta_q(x)$ , solution of  $y(qx) = y(x) + 1$ ,

to write a meromorphic fundamental solution to any  $q$ -difference system that is regular singular system at 0 or at  $\infty$  (see [Sau00, §0.1]).

*Remark 1.16.* If  $K$  is a finitely generated extension of  $\mathbb{Q}$  and  $q$  is not a root of unity, one can always embed  $K$  in  $\mathbb{C}$  or in  $\mathbb{C}_p$  in a way that  $|q| \neq 1$ . Let us focus on the case of an embedding in  $\mathbb{C}_p$ . Since the Jacobi Theta function converges over  $\mathbb{C}_p^*$ , one can transpose the results of J. Sauloy and T. Dreyfus to the  $p$ -adic setting and construct a fundamental matrix of solutions of  $p$ -adic meromorphic function over  $\mathbb{C}_p^*$  for a linear  $q$ -difference system defined over  $K(x)$ . In other words, one can always assume that a system with coefficients in  $K(x)$  has a fundamental matrix of solutions with meromorphic coefficients in some sense, archimedean or  $p$ -adic. This result is commonly accepted but yet there are no references for it. It would allow us to apply the results below to a broader range of cases.

Theorem 1.14 by Praagman ensures the existence of nice solutions, but it has a cost. The field of meromorphic functions over  $\mathbb{C}^*$  fixed by  $\sigma_q$  coincides with the field of meromorphic functions over the torus  $\mathbb{C}^*/q^{\mathbb{Z}}$ , therefore we have enlarged considerably the field of constants with respect to the “expected” algebraically closed field of constants  $\mathbb{C}$ .

Since  $\sigma_q$  and  $\partial := \frac{d}{dx}$  commute, the derivation  $\partial$  stabilizes  $\mathbb{C}_E$  inside  $\mathcal{M}(\mathbb{C}^*)$ , so that  $\mathbb{C}_E$  is naturally endowed with a structure of  $\partial$ -field. It is a trivial  $\sigma_q$ -field. Let  $\overline{\mathbb{C}}_E$  be an algebraic closure of  $\mathbb{C}_E$  and  $\widetilde{\mathbb{C}}_E$  a differential closure of  $\overline{\mathbb{C}}_E$  with respect to  $\partial$  (cf. [CS06, §9.1]). We still denote by  $\partial$  the derivation of  $\widetilde{\mathbb{C}}_E$ . One endows  $\widetilde{\mathbb{C}}_E$  and  $\overline{\mathbb{C}}_E$  with a structure of trivial  $\sigma_q$ -field. We want to show that these trivial  $\sigma_q$ -fields are *compatible* with the  $\sigma_q$ -field  $\mathbb{C}(x)$  in the sense that there exists a joint  $\sigma_q$ -field extension of  $\mathbb{C}(x)$  and  $\widetilde{\mathbb{C}}_E$  (see Corollary 1.19 below).

**Lemma 1.17.** *Let  $\mathcal{C}$  be a field extension of  $\mathbb{C}$  endowed with a trivial action of  $\sigma_q$ . The fields  $\mathcal{C}$  and  $\mathbb{C}(x)$  are linearly disjoint in  $\mathcal{C}(x)$  over  $\mathbb{C}$ .*

*Remark 1.18.* The inclusion  $\mathbb{C} \hookrightarrow \mathcal{C}$  extends to an inclusion of field of rational functions  $\mathbb{C}(x) \hookrightarrow \mathcal{C}(x)$ , therefore the statement above makes sense.

*Proof.* This is a well known property of difference fields and the proof uses very standard ideas. We give it here for completeness. Let  $f_0, \dots, f_r \in \mathcal{C}$  be linearly independent over  $\mathbb{C}$  and let us suppose that they become linearly dependent over  $\mathbb{C}(x)$ . We suppose that  $r > 0$  is minimal for this property. Then there exist  $a_1, \dots, a_r \in \mathbb{C}(x) \setminus \{0\}$ , not all belonging to  $\mathbb{C}$ , such that  $f_0 + a_1 f_1 + \dots + a_r f_r = 0$ . Applying  $\sigma_q$  and subtracting the obtained equation, we deduce that  $(\sigma_q(a_1) - a_1)f_1 + \dots + (\sigma_q(a_r) - a_r)f_r = 0$ . The minimality of  $r$  implies that the  $a_i$ 's are in  $\mathbb{C}$ , against the assumption. This proves the claim.  $\square$

Let  $\mathcal{C}$  be a trivial  $\sigma_q$ -field extension of  $\mathbb{C}$ . Thanks to the previous lemma, we know the compositum  $\mathcal{C}(x)$  of  $\mathcal{C}$  and  $\mathbb{C}(x)$  over  $\mathbb{C}$  coincides with the field of fractions of  $\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x)$ . We have:

**Corollary 1.19.** *Let  $\mathcal{C}$  be a  $(\sigma_q, \partial)$ -field extension of  $\mathbb{C}$  and a trivial  $\sigma_q$ -field. The field  $\mathcal{C}(x)$  is a  $(\sigma_q, \partial)$ -field with the action of  $\sigma_q$  defined by the properties that  $\sigma_q|_{\mathcal{C}}$  is the identity of  $\mathcal{C}$  and  $\sigma_q(x) = qx$ .*

*Proof.* The Leibnitz rule allows to extend  $\partial$  to the  $\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x)$  and one can easily extend  $\partial$  to its quotient ([vdPS03, Exercices 1.5]). The action of  $\sigma_q$  on  $\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x)$  defined by  $id \otimes \sigma_q$  is injective and extends to  $\mathcal{C}(x)$ . The commutativity of  $\sigma_q$  and  $\partial$  is straightforward.  $\square$

Corollary 1.19, taking  $\mathcal{C} = \mathbb{C}_E$ , allows to consider the  $(\sigma_q, \partial)$ -field extensions  $\mathbb{C}_E(x)$  of  $\mathbb{C}(x)$ . We can finally construct a weak parameterized Picard-Vessiot ring associated with Praagman's meromorphic solutions:

**Proposition 1.20.** *Let  $\sigma_q(\vec{y}) = A\vec{y}$ , with  $A \in \mathrm{GL}_\nu(\mathbb{C}(x))$ , be a  $q$ -difference system and let  $U \in \mathrm{GL}_\nu(\mathcal{M}(\mathbb{C}^*))$  be a fundamental solution matrix. The ring  $R_E := \mathbb{C}_E(x)\{U, \det U^{-1}\}_\partial$  is a weak parameterized Picard-Vessiot ring over  $\mathbb{C}_E(x)$  for  $\sigma_q(\vec{y}) = A\vec{y}$  and is an integral domain.*

*Proof.* It is enough to notice that  $R_E \subset \mathcal{M}(\mathbb{C}^*)$  and that  $\mathbb{C}_E \subset R_E^{\sigma_q} \subset \mathcal{M}(\mathbb{C}^*)^{\sigma_q} = \mathbb{C}_E$ .  $\square$

### 1.3 Properties of the weak Picard-Vessiot rings $R$ , $R_E$ and $\tilde{R}$

Let  $\sigma_q(\vec{y}) = A\vec{y}$ , with  $A \in \mathrm{GL}_\nu(\mathbb{C}(x))$ , be a  $q$ -difference system. We have constructed three weak parameterized Picard-Vessiot rings for  $\sigma_q(\vec{y}) = A\vec{y}$ :

1. the weak parameterized Picard-Vessiot ring  $R$  over  $\mathbb{C}(x)$ , which is  $\sigma_q$ -simple and satisfies  $R^{\sigma_q} = \mathbb{C}$ , constructed applying Proposition 1.8 to  $\sigma_q(\vec{y}) = A\vec{y}$ , seen as a system defined over  $\mathcal{F} = \mathbb{C}(x)$ ;
2. the weak parameterized Picard-Vessiot ring  $R_E$  over  $\mathbb{C}_E(x)$ , constructed in Proposition 1.20;
3. the (weak) parameterized Picard-Vessiot ring  $\tilde{R}$  over  $\tilde{\mathbb{C}}_E(x)$ , constructed applying Proposition 1.6 to  $\sigma_q(\vec{y}) = A\vec{y}$ , seen as a system defined over  $\mathcal{F} = \tilde{\mathbb{C}}_E(x)$ .

We remind that  $R$  can be written in the form

$$(1.3) \quad R = \mathbb{C}(x)\{Y, \det Y^{-1}\}_\partial / \mathfrak{q},$$

where  $Y$  is an invertible matrix satisfying the system  $\sigma_q(\vec{y}) := A\vec{y}$  and  $\mathfrak{q}$  is not only a maximal  $(\sigma_q, \partial)$ -ideal but also a maximal  $\sigma_q$ -ideal (since  $R$  is  $\sigma_q$ -simple, after Proposition 1.8).

Definition 1.10 applied to the three settings above allows to define the group schemes  $G_R^\partial$ ,  $G_{R_E}^\partial$  and  $G_{\tilde{R}}^\partial$ , respectively. As functors they are represented by a  $\partial$ - $\mathbb{C}$ -subgroup scheme of  $\mathrm{GL}_\nu(\mathbb{C})$  (resp. a  $\partial$ - $\mathbb{C}_E$ -subgroup scheme of  $\mathrm{GL}_\nu(\mathbb{C}_E)$ , a  $\partial$ - $\tilde{\mathbb{C}}_E$ -subgroup scheme of  $\mathrm{GL}_\nu(\tilde{\mathbb{C}}_E)$ ). See Proposition 1.11. We will prove that they become isomorphic after a convenient field extension. To do so, we need to prove some properties of the three Picard-Vessiot rings above and to give a Tannakian description of each one of the three groups  $G_R^\partial$ ,  $G_{R_E}^\partial$  and  $G_{\tilde{R}}^\partial$ . It will be the object of §2 and §3.

The following statement is differential analogue of [CHS08, Proposition 2.4].

**Proposition 1.21.** *Let  $\sigma_q(\vec{y}) = A\vec{y}$  be a  $q$ -difference system defined over  $\mathbb{C}(x)$ . Let  $\mathcal{F}$  be a  $(\sigma_q, \partial)$ -field extension of  $\mathbb{C}(x)$  of the form  $\mathcal{C}(x)$ , where  $\mathcal{C}$  a  $(\sigma_q, \partial)$ -field extension of  $\mathbb{C}$ , which is a trivial  $\sigma_q$ -field (for instance  $\mathcal{F} = \mathbb{C}_E(x)$  or  $\tilde{\mathbb{C}}_E(x)$ ). In the notation of Eq. (1.3),  $S := \mathcal{F}\{Y, \det Y^{-1}\}_\partial / \mathfrak{q}\mathcal{F}$  is a parameterized Picard-Vessiot ring for  $\sigma_q(\vec{y}) = A\vec{y}$  over  $\mathcal{F}$  and  $S^{\sigma_q} = \mathcal{C}$ .*

*Proof.* First we remark that  $\mathfrak{q}\mathcal{F} \subsetneq \mathcal{F}\{Y, \det Y^{-1}\}_\partial$  and hence that  $S$  is non-zero. Indeed, if  $1 = \sum_{i \in I} \lambda_i P_i$  with  $P_i \in \mathfrak{q}$  and  $\lambda_i \in \mathcal{F}$ , it is enough to expand the  $\lambda_i$ 's in a  $\mathbb{C}(x)$ -basis of  $\mathcal{F}$  to conclude that  $1 \in \mathfrak{q}$ .

We consider the natural map of  $\sigma_q$ -rings

$$\phi : R \otimes_{\mathbb{C}} \mathcal{C} \rightarrow S.$$

We want to prove that  $\phi$  is injective. Since  $\phi(1 \otimes 1) = 1$  and  $S \neq 0$ ,  $\mathfrak{J} := \text{Ker } \phi$  is a proper  $\sigma_q$ -ideal of  $R \otimes_{\mathbb{C}} \mathcal{C}$ . Proposition 1.8 implies that  $R$  is  $\sigma_q$ -simple. Moreover  $R^{\sigma_q} = \mathbb{C}$ . Therefore the  $\sigma_q$ -ideal  $\mathfrak{J}$  in  $R \otimes_{\mathbb{C}} \mathcal{C}$  is generated by  $\mathfrak{J} \cap R$  (see [vdPS97, Lemma 1.11]). We deduce that  $\mathfrak{J}$  is  $\{0\}$ . This means that  $\phi$  is injective. Notice that the same argument shows that any  $\sigma_q$ -ideal  $\mathfrak{J}$  of  $R \otimes_{\mathbb{C}} \mathcal{C}$  is generated by its intersection with  $R$ . Since  $R$  is  $\sigma_q$ -simple, we deduce that  $R \otimes_{\mathbb{C}} \mathcal{C}$  is  $\sigma_q$ -simple.

Now let  $R' = \phi(R \otimes_{\mathbb{C}} \mathcal{C})$ . Since  $\phi$  is a  $\sigma_q$ -morphism, the ring  $R'$  is  $\sigma_q$ -simple. Lemma 1.17 implies that the field  $\mathcal{F}$  is the fraction field of  $\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x)$ . Then, it is easily seen that for any  $P \in \mathcal{F}\{Y, \det Y^{-1}\}_\partial$  there exist  $a \in (\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x))^*$  such that  $aP \in \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x)\{Y, \det Y^{-1}\}_\partial$ . Then, for all  $x \in S$ , there exists  $a \in (\mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(x))^*$  such that  $ax \in R'$ . This proves that any  $\sigma_q$ -ideal  $\mathfrak{J}$  in  $S$  is generated by  $\mathfrak{J} \cap R'$ , hence  $S$  is  $\sigma_q$ -simple and thus  $(\sigma_q, \partial)$ -simple. We conclude that  $S$  is a parameterized Picard-Vessiot ring for  $\sigma_q(\vec{y}) = A\vec{y}$ .

Finally, for any  $c \in S^{\sigma_q}$ , the set  $\mathfrak{h} = \{a \in R' | ac \in R'\}$  is a non-zero  $\sigma_q$ -ideal of  $R'$ . Since  $R'$  is  $\sigma_q$ -simple, we get that  $1 \in \mathfrak{h}$ . Therefore  $c \in R'$  and  $S^{\sigma_q} = (R')^{\sigma_q} = \phi(R \otimes_{\mathbb{C}} \mathcal{C})^{\sigma_q}$ . If one considers a  $\mathbb{C}$ -basis of  $\mathcal{C}$ , which is formed by  $\sigma_q$ -constants, one can easily prove that the  $\sigma_q$ -constants of  $R \otimes_{\mathbb{C}} \mathcal{C}$  coincide with  $R^{\sigma_q} \otimes_{\mathbb{C}} \mathcal{C} = \mathcal{C}$  since  $R^{\sigma_q} = \mathbb{C}$ .  $\square$

As corollary of the previous proposition, we find:

**Corollary 1.22.** *Let  $\sigma_q(\vec{y}) = A\vec{y}$  be a  $q$ -difference system defined over  $\mathbb{C}(x)$  and let  $R, R_E$  and  $\tilde{R}$  be the weak parameterized Picard-Vessiot rings attached to  $\sigma_q(\vec{y}) = A\vec{y}$ . As above, we write  $R = \mathbb{C}(x)\{Y, \det Y^{-1}\}_\partial / \mathfrak{q}$ . We consider the two rings:*

$$S := \mathbb{C}_E(x)\{Y, \det Y^{-1}\}_\partial / \mathfrak{q}\mathbb{C}_E(x). \quad \text{and} \quad \tilde{S} := \tilde{\mathbb{C}}_E(x)\{Y, \det Y^{-1}\}_\partial / \mathfrak{q}\tilde{\mathbb{C}}_E(x)$$

Then the two natural maps

$$S \otimes_{\mathbb{C}_E} \tilde{\mathbb{C}}_E \longrightarrow R_E \otimes_{\mathbb{C}_E} \tilde{\mathbb{C}}_E \quad \text{and} \quad \tilde{S} \longrightarrow \tilde{R}$$

are both an isomorphism of  $(\sigma_q, \partial)$ - $\tilde{\mathbb{C}}_E(x)$ -algebras.

*Proof.* By Proposition 1.21, applied to  $\mathcal{F} = \mathbb{C}_E(x)$  and  $\mathcal{F} = \tilde{\mathbb{C}}_E(x)$ , we find that  $S$  (resp.  $\tilde{S}$ ) is a parameterized Picard-Vessiot ring for  $\sigma_q(\vec{y}) = A\vec{y}$  over  $\mathbb{C}_E(x)$  (resp.  $\tilde{\mathbb{C}}_E(x)$ ) such that  $S^{\sigma_q} = \mathbb{C}_E$  (resp.  $\tilde{S}^{\sigma_q} = \tilde{\mathbb{C}}_E$ ). Since  $\tilde{\mathbb{C}}_E$  is differentially closed, Proposition 1.6 assures that two parameterized Picard-Vessiot rings for the same  $q$ -difference system over  $\tilde{\mathbb{C}}_E(x)$  are isomorphic as  $\tilde{\mathbb{C}}_E(x)$ - $(\sigma_q, \partial)$ -algebras. The first isomorphism follows from this fact.

The second isomorphism comes from a parameterized version of [CHS08, Proposition 2.7]. Its proof follows line by line the proof in the algebraic case, but we give it here for sake of completeness. Let us denote by  $F_E$  the fraction field of  $R_E$  and let  $X = (X_{i,j})$  be a  $\nu \times \nu$ -matrix of differential indeterminates over  $F_E$ . Let  $\mathcal{S} := \mathbb{C}_E(x)\{X, \det X^{-1}\}_\partial \subset F_E\{X, \det X^{-1}\}_\partial$ . Define a  $(\sigma_q, \partial)$ -structure on  $F_E\{X, \det X^{-1}\}_\partial$  by setting  $\sigma_q(X) := AX$ ,  $\sigma_q(\partial X) := A\partial X + \partial AX$ , and so on. This induces a  $(\sigma_q, \partial)$ -structure on  $\mathcal{S}$ . Since  $S$  is a parameterized Picard-Vessiot ring for  $\sigma_q(Y) = AY$  view over  $\mathbb{C}_E(x)$ , we can write  $S = \mathcal{S}/\mathfrak{p}$ , where  $\mathfrak{p}$  is a maximal  $(\sigma_q, \partial)$ -ideal of  $\mathcal{S}$ . Now, let  $U \in \text{GL}_\nu(R_E)$  be fundamental solution matrix of  $\sigma_q(Y) = AY$ . Define  $Y = (Y_{i,j}) \in \text{GL}_\nu(F_E\{X, \det X^{-1}\}_\partial)$  via  $Y := U^{-1}X$  and remark that  $\sigma_q(Y) = Y$  and  $F_E\{X, \det X^{-1}\}_\partial = F_E\{Y, \det Y^{-1}\}_\partial$ . Define  $\mathcal{S}_1 := \mathbb{C}_E\{Y, \det Y^{-1}\}_\partial$ . The ideal  $\mathfrak{p} \subset \mathcal{S} \subset F_E\{X, \det X^{-1}\}_\partial$  generates a  $(\sigma_q, \partial)$ -ideal  $(\mathfrak{p})$  in  $F_E\{X, \det X^{-1}\}_\partial$ , which intersected with  $\mathcal{S}_1$  gives a  $\partial$ -ideal  $\mathfrak{a}$ . Since  $\tilde{\mathbb{C}}_E$  is differentially closed and  $\mathcal{S}_1/\mathfrak{a}$  is differentially finitely generated over  $\mathbb{C}_E$ , we

find a differential homomorphism  $\mathcal{S}_1 \rightarrow \mathcal{S}_1/\mathfrak{a} \rightarrow \tilde{\mathcal{C}}_E$ . We can extend this homomorphism into a  $(\sigma_q, \partial)$ -morphism  $F_E\{X, \det X^{-1}\}_\partial = F_E \otimes \mathcal{S}_1 \rightarrow F_E \otimes_{\mathbb{C}_E} \tilde{\mathcal{C}}_E$ . Restricted to  $\mathcal{S}$ , we find a  $(\sigma_q, \partial)$ -morphism  $\mathcal{S} \rightarrow F_E \otimes \tilde{\mathcal{C}}_E$ , whose kernel contains  $\mathfrak{p}$ . By maximality of  $\mathfrak{p}$ , we have equality and we get an embedding  $\iota: S = \mathcal{S}/\mathfrak{p} \rightarrow F_E \otimes \tilde{\mathcal{C}}_E$ . Now, if we denote by  $V \in \mathrm{GL}_\nu(S)$  a fundamental solution matrix of  $\sigma_q(Y) = AY$  and we remind that  $(F_E \otimes \tilde{\mathcal{C}}_E)^{\sigma_q} = \tilde{\mathcal{C}}_E$ , we find that  $\iota(V) = UD$  with  $D \in \mathrm{GL}_\nu(\tilde{\mathcal{C}}_E)$ . Since  $S$  (resp.  $R_E$ ) is differentially generated over  $\mathbb{C}_E(x)$  (resp.  $\tilde{\mathcal{C}}_E(x)$ ) by  $V$  (resp.  $U$ ) and the inverse of its determinant, this allows us to conclude that  $\iota(S \otimes \tilde{\mathcal{C}}_E) = R_E \otimes \tilde{\mathcal{C}}_E$ .  $\square$

## 2 The category of $q$ -difference modules

We are interested in giving an interpretation of Picard-Vessiot extensions from a categorical point of view, therefore we introduce here the category of  $q$ -difference modules. Since we are interested in studying the action of the derivation  $\partial$ , we will quickly review the basic definitions and properties of differential Tannakian categories, introduced by Kamensky [Kam10] and Ovchinnikov in [Ovc09].

### 2.1 $q$ -difference modules

Let  $(\mathcal{F}, \sigma_q, \partial)$  be a  $(\sigma_q, \partial)$ -field of characteristic zero and let  $\mathcal{C} = \mathcal{F}^{\sigma_q}$ .

**Definition 2.1.** A  $q$ -difference module  $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$  (of rank  $\nu$ ) over  $\mathcal{F}$  is a finite dimensional  $\mathcal{F}$ -vector space  $M_{\mathcal{F}}$  (of dimension  $\nu$ ) equipped with an invertible  $\sigma_q$ -semi-linear operator  $\Sigma_q: M_{\mathcal{F}} \rightarrow M_{\mathcal{F}}$ , i.e., a bijective additive map from  $M_{\mathcal{F}}$  to itself such that

$$\Sigma_q(fm) = \sigma_q(f)\Sigma_q(m), \text{ for any } f \in \mathcal{F} \text{ and } m \in M_{\mathcal{F}}.$$

We will call  $\Sigma_q$  a  $q$ -difference operator over  $M_{\mathcal{F}}$  or the  $q$ -difference operator of  $\mathcal{M}_{\mathcal{F}}$ .

A  $q$ -difference submodule  $\mathcal{N}_{\mathcal{F}}$  of  $\mathcal{M}_{\mathcal{F}}$  is a  $\mathcal{F}$ -vector subspace  $N_{\mathcal{F}}$  of  $M_{\mathcal{F}}$  that is set-wise invariant with respect to  $\Sigma_q$ . Then,  $\mathcal{N}_{\mathcal{F}} = (N_{\mathcal{F}}, \Sigma_q|_{N_{\mathcal{F}}})$  is a  $q$ -difference module.

A morphism of  $q$ -difference modules (over  $\mathcal{F}$ ) is a morphism of the underlying  $\mathcal{F}$ -vector spaces, commuting with the  $q$ -difference operators defined on the domain and on the image of the morphism. We denote by  $\mathrm{DiffMod}(\mathcal{F}, \sigma_q)$  the category of  $q$ -difference modules over  $\mathcal{F}$ .

Let  $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$  be a  $q$ -difference module over  $\mathcal{F}$  of rank  $\nu$ . We fix a basis  $\underline{e}$  of  $M_{\mathcal{F}}$  over  $\mathcal{F}$ . Let  $A \in \mathrm{GL}_\nu(\mathcal{F})$  be such that:

$$\Sigma_q \underline{e} = \underline{e}A.$$

If  $\underline{f}$  is another basis of  $M_{\mathcal{F}}$ , such that  $\underline{f} = \underline{e}F$ , with  $F \in \mathrm{GL}_\nu(\mathcal{F})$ , then  $\Sigma_q \underline{f} = \underline{f}B$ , with  $B = F^{-1}A\sigma_q(F)$ . Conversely, given an invertible matrix  $A \in \mathrm{GL}_\nu(\mathcal{F})$ , one constructs a  $q$ -difference module  $\mathcal{M}_{\mathcal{F}}$  as follows:  $M_{\mathcal{F}} = \mathcal{F}^\nu$  and  $\Sigma_q \underline{e} = \underline{e}A$  with  $\underline{e}$  the canonical basis of  $\mathcal{F}^\nu$ .

The elements  $m \in M_{\mathcal{F}}$  such that  $\Sigma_q(m) = m$  are called horizontal. If a horizontal element  $m$  corresponds to a vector  $\vec{y} \in \mathcal{F}^\nu$  with respect to the basis  $\underline{e}$ , we have:  $\underline{e}\vec{y} = \Sigma_q(\underline{e}\vec{y}) = \underline{e}A\sigma_q(\vec{y})$ . Therefore  $\vec{y}$  verifies the linear  $q$ -difference system  $\sigma_q(\vec{y}) = A^{-1}\vec{y}$ , that we call the linear difference system associated to  $\mathcal{M}_{\mathcal{F}}$  with respect to the basis  $\underline{e}$ .

The constructions of linear algebra (i.e., direct sums, duals, the tensor products) of the underlying vector spaces of two  $q$ -difference modules over  $\mathcal{F}$  can be endowed with a structure of  $q$ -difference modules (see for instance [vdPS97, Chapter 12], [DV02, Part I] or [DVRSZ03]). The category  $\mathrm{DiffMod}(\mathcal{F}, \sigma_q)$  is a tensor category and we denote by  $\mathbf{1} = (\mathcal{F}, \sigma_q)$  the unit object for the tensor product. It is also a rigid category, i.e., it has internal Homs and each object is canonically isomorphic to its bidual. It is therefore a Tannakian category in the sense of [DMOS82] (see [Sau03]). If  $\mathcal{C}$  is algebraically closed, the general theory of Tannakian categories ensures that it is equivalent to the category of representation of a certain  $\mathcal{C}$ -group-scheme  $G$ .

## 2.2 The differential Tannakian structure of $\text{DiffMod}(\mathcal{F}, \sigma_q)$

In this section we define the prolongation functor in the general framework of projective modules. The definition may seem very abstract at the first glance but we will show in Example 2.4 that it is an incarnation of Remark 1.2.

We consider a  $\partial$ -field  $k$  and a  $\partial$ - $k$ -algebra  $\mathcal{S}$ . We denote by  $\mathcal{S}[\partial]_{\leq 1}$  the 2-dimensional free  $\mathcal{S}$ -module of differential operators of order less or equal to 1. In agreement with the Leibniz rule, the right  $\mathcal{S}$ -module structure of  $\mathcal{S}[\partial]$  is given by  $\partial \cdot a = a \cdot \partial + \partial(a)$ .

**Definition 2.2.** We define on the category  $\text{Proj}_{\mathcal{S}}$  of finitely generated projective modules over  $\mathcal{S}$  an endofunctor  $F_{\partial}$ , called prolongation functor, as follows:

- For  $M$  an object of  $\text{Proj}_{\mathcal{S}}$ , we define  $F_{\partial}(M) := \mathcal{S}[\partial]_{\leq 1} \otimes_{\mathcal{S}} M$ , where the tensor product is considered with respect to the right  $\mathcal{S}$ -module structure of  $\mathcal{S}[\partial]_{\leq 1}$ . The  $\mathcal{S}$ -module structure of  $F_{\partial}(M)$  is defined by:  $\lambda(\partial \otimes v) = \partial \otimes \lambda v - \partial(\lambda) \otimes v$ , for all  $\lambda \in \mathcal{S}$  and  $v \in M$ , and extended by linearity.
- If  $f \in \text{Hom}_{\text{Proj}_{\mathcal{S}}}(M, N)$ , we define  $F_{\partial}(f) : F_{\partial}(M) \rightarrow F_{\partial}(N)$  as:  $F_{\partial}(f)(\partial^i \otimes m) = \partial^i \otimes f(m)$ , for  $i = 0, 1$ , where we have used the convention that  $\partial^0$  is the identity map.

*Remark 2.3.* We will informally call constructions of linear differential algebra the family of all the constructions of linear algebra plus  $F_{\partial}$ . Notice that, if  $\partial$  is the trivial derivation, then  $F_{\partial}(M)$  coincides with the direct sum  $M \oplus M$ .

The underlying vector spaces of the objects of  $\text{DiffMod}(\mathcal{F}, \sigma_q)$  form a subcategory of  $\text{Proj}_{\mathcal{F}}$ . Since  $\mathcal{F}$  is a field,  $\text{Proj}_{\mathcal{F}}$  is the category of vector spaces over  $\mathcal{F}$ , that we will also denote  $\text{Vect}_{\mathcal{F}}$ . Given an object  $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$  of  $\text{DiffMod}(\mathcal{F}, \sigma_q)$ , we are able to extend the action of  $\Sigma_q$  to  $F_{\partial}(M_{\mathcal{F}})$  via

$$\Sigma_q(\partial^i(m)) := \partial^i(\Sigma_q(m)), \text{ for } i = 0, 1 \text{ and } m \in M_{\mathcal{F}}.$$

We set  $F_{\partial}(\mathcal{M}_{\mathcal{F}}) = (F_{\partial}(M_{\mathcal{F}}), \Sigma_q)$ . This shows that  $F_{\partial}$  extends to an endofunctor of  $\text{DiffMod}(\mathcal{F}, \sigma_q)$ . Together with this additional structure,  $(\text{DiffMod}(\mathcal{F}, \sigma_q), F_{\partial})$  is a differential Tannakian category over  $\mathcal{C}$  as defined in [GGO13, §4.4], i.e., a  $\mathcal{C}$ -linear, tensor, rigid category together with a prolongation functor, satisfying some natural commutative diagrams, that we are not recalling here.

**Example 2.4.** Let  $\mathcal{M}_{\mathcal{F}} = (M_{\mathcal{F}}, \Sigma_q)$  be a  $q$ -difference module over  $\mathcal{F}$ . We fix a basis  $\underline{e} = (e_1, \dots, e_{\nu})$  of  $M_{\mathcal{F}}$  such that  $\Sigma_q \underline{e} = \underline{e}A$ , for some  $A \in \text{GL}_{\nu}(\mathcal{F})$ . A basis of  $F_{\partial}(M_{\mathcal{F}})$  is given by  $(\underline{e}, \partial \otimes \underline{e})$ . The definition of  $\Sigma_q$  on  $F_{\partial}(M_{\mathcal{F}})$  is reminiscent of Remark 1.2:

$$\Sigma_q(\underline{e}, \partial \otimes \underline{e}) = (\underline{e}, \partial \otimes \underline{e}) \begin{pmatrix} A & \partial(A) \\ 0 & A \end{pmatrix}.$$

Following [GGO13, Definition 4.9], we recall the notion of differential fiber functor.

**Definition 2.5.** Let  $\mathcal{S}$  be a  $\partial$ - $\mathcal{C}$ -algebra. We say that a functor  $\omega : \text{DiffMod}(\mathcal{F}, \sigma_q) \rightarrow \text{Proj}_{\mathcal{S}}$  is a differential fiber functor over  $\mathcal{S}$  if it is exact, faithful,  $\mathcal{C}$ -linear, tensor compatible and if it commutes to  $F_{\partial}$ , i.e., if  $F_{\partial} \circ \omega = \omega \circ F_{\partial}$  as a natural isomorphism. We say that  $\omega$  is a neutral differential fiber functor if  $\mathcal{S} = \mathcal{C}$ .

*Remark 2.6.* For further reference we point out that:

- A differential fiber functor is also a fiber functor for the classical Tannakian theory [DMOS82, p. 148].
- The forgetful functor  $\eta_{\mathcal{F}} : \text{DiffMod}(\mathcal{F}, \sigma_q) \rightarrow \text{Vect}_{\mathcal{F}}$ , which assigns to any  $q$ -difference module its underlying  $\mathcal{F}$ -vector space, is a differential fiber functor over  $\mathcal{F}$ .

Since one of our main purposes is to compare distinct fiber functors, we introduce the functor of differential tensor morphisms between two differential fiber functors.

**Definition 2.7** (Def. 1.12 in [DMOS82]). Let  $\omega_1, \omega_2 : \text{DiffMod}(\mathcal{F}, \sigma_q) \rightarrow \text{Proj}_{\mathcal{S}}$  be two differential fiber functors. For any  $\mathcal{S}$ -algebra  $\mathcal{R}$ , we define  $\text{Hom}^{\otimes}(\omega_1, \omega_2)(\mathcal{R})$  as the set of all sequences of the form  $\{\lambda_{\mathcal{X}_{\mathcal{F}}} | \mathcal{X}_{\mathcal{F}} \text{ object of } \text{DiffMod}(\mathcal{F}, \sigma_q)\}$ , such that:

- $\lambda_{\mathcal{X}_{\mathcal{F}}}$  is an  $\mathcal{R}$ -linear homomorphism from  $\omega_1(\mathcal{X}_{\mathcal{F}}) \otimes \mathcal{R}$  to  $\omega_2(\mathcal{X}_{\mathcal{F}}) \otimes \mathcal{R}$ ,
- $\lambda_{\mathbf{1}}$  is the identity on  $\mathbf{1} \otimes \mathcal{R}$ ,
- for every  $\alpha \in \text{Hom}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$ , we have  $\lambda_{\mathcal{Y}_{\mathcal{F}}} \circ (\omega_1(\alpha) \otimes id_{\mathcal{R}}) = (\omega_2(\alpha) \otimes id_{\mathcal{R}}) \circ \lambda_{\mathcal{X}_{\mathcal{F}}}$ ,
- $\lambda_{\mathcal{X}_{\mathcal{F}}} \otimes \lambda_{\mathcal{Y}_{\mathcal{F}}} = \lambda_{\mathcal{X}_{\mathcal{F}} \otimes \mathcal{Y}_{\mathcal{F}}}$ .

For a  $\partial$ - $\mathcal{S}$ -algebra  $\mathcal{R}$ , we define  $\text{Hom}^{\otimes, \partial}(\omega_1, \omega_2)(\mathcal{R})$  as the subset of  $\text{Hom}^{\otimes}(\omega_1, \omega_2)(\mathcal{R})$  of all sequences such that  $F_{\partial}(\lambda_{\mathcal{X}_{\mathcal{F}}}) = \lambda_{F_{\partial}(\mathcal{X}_{\mathcal{F}})}$ , where the  $F_{\partial}$  on the left hand side is the prolongation functor on  $\text{Proj}_{\mathcal{R}}$  whereas the  $F_{\partial}$  on the right hand side is the prolongation functor in  $\text{DiffMod}(\mathcal{F}, \sigma_q)$  (see [Ovc09, §4.3]).

The functor  $\text{Hom}^{\otimes, \partial}(\omega_1, \omega_2)$ , composed with the forgetful functor from  $\partial$ - $\mathcal{S}$ -algebras to  $\mathcal{S}$ -algebras is a subfunctor of  $\text{Hom}^{\otimes}(\omega_1, \omega_2)$ . By [Del90, Prop.6.6] the functor  $\text{Hom}^{\otimes}(\omega_1, \omega_2)$  is representable by a  $\mathcal{S}$ -scheme.

Since morphisms of tensor functors are isomorphisms (see [DMOS82, Proposition 1.13]), differential morphisms of differential tensor functors are also differential isomorphisms. Thus, we will now write  $\text{Isom}^{\otimes, \partial}(\omega_1, \omega_2)$  (resp.  $\text{Isom}^{\otimes}(\omega_1, \omega_2)$ ) instead of  $\text{Hom}^{\otimes, \partial}(\omega_1, \omega_2)$  (resp.  $\text{Hom}^{\otimes}(\omega_1, \omega_2)$ ) and, when  $\omega_1 = \omega_2 = \omega$ , we write  $\text{Aut}^{\otimes, \partial}(\omega)$  (resp.  $\text{Aut}^{\otimes}(\omega)$ ). In that special case, it occurs that the functor  $\text{Aut}^{\otimes, \partial}(\omega)$  (resp.  $\text{Aut}^{\otimes}(\omega)$ ) is a group functor, where the composition is given by the composition of morphisms.

We rephrase [GGO13, Proposition 4.25] in our setting:

**Proposition 2.8.** *Let  $\mathcal{S}$  be a  $\partial$ - $\mathcal{C}$ -algebra and let  $\omega : \text{DiffMod}(\mathcal{F}, \sigma_q) \rightarrow \text{Proj}_{\mathcal{S}}$  be a differential fiber functor. Let  $A$  be the  $\mathcal{S}$ -Hopf algebra that represents the functor  $\text{Aut}^{\otimes}(\omega)$  (see [Del90, Proposition 6.19]). Then,  $A$  has a canonical structure of  $\partial$ - $\mathcal{S}$ -Hopf algebra and represents the functor  $\text{Aut}^{\otimes, \partial}(\omega)$ .*

Proposition 2.8 shows that the functor  $\text{Aut}^{\otimes, \partial}(\omega)$  is a  $\partial$ -group-scheme in the sense of Appendix B. If  $\mathcal{S}$  is a  $\partial$ -closed field extension of  $\mathcal{C}$  then one can identify  $\text{Aut}^{\otimes, \partial}(\omega)(\mathcal{S})$  with a subgroup of  $\text{GL}_{\nu}(\mathcal{S})$  defined as the zero set of polynomial differential equations with coefficients in  $\mathcal{S}$ .

### 3 Galois groups

Let  $(F, \sigma_q, \partial)$  be a  $(\sigma_q, \partial)$ -field. We fix a  $q$ -difference module  $\mathcal{M}_{\mathcal{F}}$  in  $\text{DiffMod}(\mathcal{F}, \sigma_q)$  and consider three categories generated by  $\mathcal{M}_{\mathcal{F}}$ . First of all, we consider the strictly full subcategory  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\oplus}$  of  $\text{DiffMod}(\mathcal{F}, \sigma_q)$ , that contains the subquotients of all finite direct sums of copies of  $\mathcal{M}_{\mathcal{F}}$ , i.e., is the abelian subcategory generated by  $\mathcal{M}_{\mathcal{F}}$ . Then we need the Tannakian category  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$  (resp. differential Tannakian category  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ ), that is the strictly full Tannakian (resp. differential Tannakian) category generated by  $\mathcal{M}_{\mathcal{F}}$ . It admits a very simple description: We consider the constructions of linear (resp. linear differential) algebra of  $\mathcal{M}_{\mathcal{F}}$ , i.e., the list of  $q$ -difference modules

$$\bigoplus_{i,j} \mathcal{M}_{\mathcal{F}}^{\otimes i} \otimes \mathcal{M}_{\mathcal{F}}^{*\otimes j} \quad \left( \text{resp.} \quad \bigoplus_{i,j,l,r,s} \mathcal{M}_{\mathcal{F}}^{\otimes i} \otimes \mathcal{M}_{\mathcal{F}}^{*\otimes j} \otimes F_{\partial}^l(\mathcal{M}_{\mathcal{F}}^{\otimes r} \otimes \mathcal{M}_{\mathcal{F}}^{*\otimes s}) \right),$$

where  $\mathcal{M}_{\mathcal{F}}^*$  denotes the dual of  $\mathcal{M}_{\mathcal{F}}$  and  $i, j$  are non-negative integers (resp.  $i, j, l, r, s$  are non-negative integers and  $F_{\partial}^l$  the  $l$ -th iterate of the prolongation functor). If we order the sub-objects of the constructions of linear (resp. linear differential) algebra of  $\mathcal{M}_{\mathcal{F}}$  with respect to the relation “be a direct summand”, then  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$  (resp.  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ ) is the filtering union of the abelian categories  $\langle \mathcal{N}_{\mathcal{F}} \rangle^{\oplus}$ , where  $\mathcal{N}_{\mathcal{F}}$  runs through the sub-objects of a construction of linear (resp. linear differential) algebra of  $\mathcal{M}_{\mathcal{F}}$ . This inductive description allows to see the Tannakian as well as the differential Tannakian equivalence as an inductive limit of Morita equivalences (see [DMOS82, Lemma 2.13]). Thus, for  $\mathcal{M}_{\mathcal{F}}$  a  $q$ -difference module and  $\omega : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial} \rightarrow \text{Proj}_{\mathcal{C}}$  a differential fiber functor, we denote by  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega|_{\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}})$  and by  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega)$  the groups of tensor and differential tensor automorphisms of  $\omega$ , respectively.

**Notation 3.1.** In the current notation, the group  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega)$  would be the group of tensor of automorphisms  $\omega$  as a fiber functor defined on the category  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ , forgetting the differential structure. Since we will never use such a group, we will make an abuse of notation writing  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega)$  for  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega|_{\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}})$ . The same abuse of notation will be applied to other groups defined later in the text below, unless the context requires more precision.

### 3.1 The forgetful functor and the intrinsic Galois groups

Following [And01], we pay particular attention to the forgetful fiber functor

$$\eta_{\mathcal{F}} : \text{DiffMod}(\mathcal{F}, \sigma_q) \rightarrow \text{Vect}_{\mathcal{F}},$$

that sends a  $q$ -difference module  $\mathcal{M}_{\mathcal{F}}$  onto its underlying vector space  $M_{\mathcal{F}}$ .

**Definition 3.2.** The intrinsic (resp. parameterized intrinsic) Galois group  $\text{Gal}(\mathcal{M}_{\mathcal{F}})$  (resp.  $\text{Gal}^{\partial}(\mathcal{M}_{\mathcal{F}})$ ) is the group  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}|_{\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}}})$  (resp.  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \eta_{\mathcal{F}|_{\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}}})$ ).

The defining equations of the intrinsic Galois groups can be read off from the form of the  $q$ -difference systems attached to  $\mathcal{M}_{\mathcal{F}}$  and its construction of linear differential algebra. Moreover, it enjoys an arithmetic description when  $K = k(q)$  with  $k$  a finitely generated extension of  $\mathbb{Q}$ . This arithmetic characterization depends on whether  $q$  is algebraic or transcendental over  $\mathbb{Q}$ . See [DVH19, Chapter 5 and §7.3] for an overview of the results on this topic. As an example, we present here only the result under the assumption that  $q$  is transcendental over  $\mathbb{Q}$ :

**Theorem 3.3** ([DVH19, Theorem 4 in the Introduction and Theorem 7.13]). *Let  $\mathcal{M}_{K(x)}$  be a  $q$ -difference module over  $K(x)$ . The parameterized intrinsic Galois group  $\text{Gal}^{\partial}(\mathcal{M}_{K(x)})$  is the smallest differential algebraic subgroup of  $\text{GL}(M_{K(x)})$ , whose specialization at  $\zeta_n$  contains the specialization of the operator  $\Sigma_q^n$  at  $\zeta_n$ , for almost all positive integer  $n$  and for a choice of a primitive  $n$ -th root of unity  $\zeta_n$ .*

For  $Y(qx) = A(x)Y(x)$ , the above theorem says roughly that the set of differential algebraic equations in  $K\{Z, \det Z^{-1}\}_{\partial}$  defining the parameterized intrinsic Galois group is generated by the ones that vanishes on the *curvatures* of the system, that is on

$$A(q^{n-1}x) \cdots A(qx)A(x)|_{q=\zeta_n},$$

for almost all positive integer  $n$  and for a choice  $\zeta_n$  of a primitive  $n$ -th root of unity.

### 3.2 Fiber functors associated with weak parameterized Picard-Vessiot extensions

In this section we show that a weak parameterized Picard-Vessiot ring naturally determines a neutral differential fiber functor. As in the theory of Tannakian categories, we expect the contrary to be also true, but the result is not included in the literature on differential Tannakian category. In the next section we will apply this construction to any of the rings listed in §1.3.

**Proposition 3.4.** *Let  $\mathcal{M}_{\mathcal{F}}$  be a  $q$ -difference module over  $\mathcal{F}$  and let  $R$  be a weak parameterized Picard-Vessiot ring for a  $q$ -difference system  $\sigma_q(Y) = AY$  associated to  $\mathcal{M}_{\mathcal{F}}$  in some fixed basis. Then,*

$$\begin{aligned} \omega_R : \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial} &\rightarrow \text{Vect}_{\mathcal{C}}, \\ \mathcal{N}_{\mathcal{F}} &\mapsto \text{Ker}(\Sigma_q - id, \mathcal{N}_{\mathcal{F}} \otimes_{\mathcal{F}} R) \end{aligned}$$

*is a neutral differential fiber functor.*

*Proof.* Let  $\sigma_q(\vec{y}) = A\vec{y}$  be the  $q$ -difference system associated to  $\mathcal{M}_{\mathcal{F}}$  in a fixed basis. We have  $R = \mathcal{F}\{Z, \det Z^{-1}\}_{\partial}$ , where  $Z \in \text{GL}_\nu(R)$  and  $\sigma_q(Z) = AZ$ . Hence the  $q$ -difference system attached to  $F_{\partial}(\mathcal{M}_{\mathcal{F}})$  is given by  $\sigma_q(Y) = \begin{pmatrix} A & \partial(A) \\ 0 & A \end{pmatrix} Y$  and a fundamental matrix is  $\begin{pmatrix} Z & \partial(Z) \\ 0 & Z \end{pmatrix}$ . Let  $i$  be a positive integer. Repeating the argument above, we can see that the  $q$ -difference module obtained from  $\mathcal{M}_{\mathcal{F}}$  iterating  $i$  times the prolongation functor is trivialized by  $R$ , i.e., admits a fundamental solution matrix with coefficients in  $R$ , and that more generally  $R$  trivializes any construction  $\mathcal{X}_{\mathcal{F}}$  of differential algebra of  $\mathcal{M}_{\mathcal{F}}$ . This comes from the fact that a  $q$ -difference system (resp. fundamental solution matrix) attached to  $\mathcal{X}_{\mathcal{F}}$  is obtained from  $A$  (resp.  $Z$ ) by the same construction of differential algebra. Then, it is clear that any sub-object  $\mathcal{N}_{\mathcal{F}}$  of  $\mathcal{X}_{\mathcal{F}}$  admits a fundamental solution matrix with coefficients in  $R$ . Thereby, for any object  $\mathcal{N}_{\mathcal{F}}$  in  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ , we find a functorial isomorphism between  $\mathcal{N}_{\mathcal{F}} \otimes_{\mathcal{F}} R$  and  $\omega_R(\mathcal{N}_{\mathcal{F}}) \otimes_{\mathcal{C}} R$ . We

deduce from this fact that  $\omega_R$  is a faithful, exact,  $\mathcal{C}$ -linear tensor functor. It is neutral because  $R^{\sigma_q} = \mathcal{C}$ . The fact that  $\omega_R$  intertwines with  $F_\partial$  corresponds exactly to the fact that a fundamental solution matrix attached to  $F_\partial(\mathcal{M}_{\mathcal{F}})$  is given by the prolongation of a fundamental solution matrix attached to  $\mathcal{M}_{\mathcal{F}}$ , as explained above.  $\square$

The following proposition, which is the parameterized analogue of [vdPS97, Theorem 1.32.2)], shows that the group  $G_R^\partial$  of functorial  $(\sigma_q, \partial)$ - $\mathcal{F}$ -automorphism of  $R = \mathcal{F}\{Z, \det Z^{-1}\}_\partial$  (see Definition 1.10) corresponds to the group of differential tensor automorphisms of the neutral differential fiber functor  $\omega_R$ , constructed in Proposition 3.4.

**Proposition 3.5.** *Let  $\mathcal{M}_{\mathcal{F}}$  be a  $q$ -difference module over  $\mathcal{F}$  and  $R$  be a weak parameterized Picard-Vessiot ring for a  $q$ -difference system attached to  $\mathcal{M}_{\mathcal{F}}$ . Then, the linear differential algebraic groups  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega_R)$  and  $G_R^\partial$  are isomorphic over  $\mathcal{C}$ .*

*Proof.* We adapt the proof of [vdPS97, Theorem 1.32] to our parameterized setting. Let  $\mathcal{S}$  be a  $\partial$ - $\mathcal{C}$ -algebra. For any  $q$ -difference module in  $\mathcal{N}_{\mathcal{F}}$  in  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ , the morphism  $\tau_{\mathcal{S}} \in G_R^\partial(\mathcal{S})$  acts on  $\mathcal{N}_{\mathcal{F}} \otimes_{\mathcal{F}} R \otimes_{\mathcal{C}} \mathcal{S}$  as  $\text{id} \otimes \tau_{\mathcal{S}}$ . Since this action commutes with  $\Sigma_q$ , it induces an action  $\tau_{\mathcal{N}_{\mathcal{F}}}$  of  $\tau_{\mathcal{S}}$  on  $\omega(\mathcal{N}_{\mathcal{F}}) \otimes \mathcal{S}$ . This defines a sequence of the form  $\{\tau_{\mathcal{N}_{\mathcal{F}}} | \mathcal{N}_{\mathcal{F}} \text{ object of } \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}\}$ . Let  $f : \mathcal{N}_{\mathcal{F}} \rightarrow \mathcal{V}_{\mathcal{F}}$  be a morphism in  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ . Then,  $f$  extends to a  $R \otimes \mathcal{S}$ -linear map  $f \otimes \text{id} : \mathcal{N}_{\mathcal{F}} \otimes_{\mathcal{F}} R \otimes_{\mathcal{C}} \mathcal{S} \rightarrow \mathcal{V}_{\mathcal{F}} \otimes_{\mathcal{F}} R \otimes_{\mathcal{C}} \mathcal{S}$ , which commutes with  $\Sigma_q$  and the action of  $G_R^\partial(\mathcal{S})$ . Thus,  $\tau_{\mathcal{V}_{\mathcal{F}}} \circ (\omega_R(f) \otimes \text{id}) = (\omega_R(f) \otimes \text{id}) \circ \tau_{\mathcal{N}_{\mathcal{F}}}$ . Since  $\tau_{\mathcal{S}}$  commutes with the derivation  $\partial$ , we have  $F_\partial(\tau_{\mathcal{N}_{\mathcal{F}}}) = \tau_{F_\partial(\mathcal{N}_{\mathcal{F}})}$ . Moreover,  $\tau_{\mathbf{1}}$  is clearly the identity. This induces a functorial group homomorphism  $\alpha(\mathcal{S}) : G_R^\partial(\mathcal{S}) \rightarrow \text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega_R)(\mathcal{S})$ .

Let us prove that  $\alpha(\mathcal{S})$  is injective. If  $\alpha(\mathcal{S})(\tau_{\mathcal{S}})$  is the identity, then in particular,  $\tau_{\mathcal{M}_{\mathcal{F}}}$  is the identity on  $\omega_R(\mathcal{M}_{\mathcal{F}})$ . Let  $(m_i)_i$  be a  $\mathcal{F}$ -basis of  $\mathcal{M}_{\mathcal{F}}$  and let  $X := (x_{j,i})_{0 \leq i, j \leq \nu}$  be an matrix in  $\text{GL}_\nu(R)$ , such that  $(\mu_i := \sum_j x_{j,i} m_j)_i$  is a  $\mathcal{C}$ -basis of  $\omega_R(\mathcal{M}_{\mathcal{F}})$ . The matrix  $X \in \text{GL}_\nu(R)$  is a fundamental solution matrix of the system associated to  $\mathcal{M}_{\mathcal{F}}$  in the basis  $(m_i)_i$ , whose coefficients generate  $R$  as  $\partial$ - $\mathcal{F}$ -algebra. Notice that  $\tau_{\mathcal{S}}(X) = X$ , since  $\tau_{\mathcal{M}_{\mathcal{F}}}$  acts as the identity on  $\omega_R(\mathcal{M}_{\mathcal{F}})$ . Therefore,  $\tau_{\mathcal{S}}$  is the identity on  $R \otimes \mathcal{S}$ , which proves that  $\alpha(\mathcal{S})$  is injective.

Conversely, let  $\tau = \{\tau_{\mathcal{N}_{\mathcal{F}}} | \mathcal{N}_{\mathcal{F}} \text{ object of } \langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}\}$  be an element of  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega_R)(\mathcal{S})$ . We want to construct an element  $\tau_{\mathcal{S}} \in G_R^\partial(\mathcal{S})$  such that  $\alpha(\mathcal{S})(\tau_{\mathcal{S}}) = \tau$ . Let us write  $R$  as  $\mathcal{F}\{X, \frac{1}{\det X}\}_\partial$ . The action of  $\tau_{\mathcal{M}_{\mathcal{F}}}$  in the  $\mathcal{S}$ -basis  $\mu_1 \otimes 1, \dots, \mu_\nu \otimes 1$  is given by an invertible matrix  $[\tau]_{\mathcal{S}} \in \text{GL}_\nu(\mathcal{S})$ . We consider the morphism  $\tau_{\mathcal{S}}$  of  $\mathcal{F}$ -algebra of  $R \otimes \mathcal{S}$  defined as follows:  $\tau_{\mathcal{S}}(X) = X \cdot [\tau]_{\mathcal{S}}$ ,  $\tau_{\mathcal{S}}(X^{(h)}) = \partial^h(X[\tau]_{\mathcal{S}})$  for any non-negative integer  $h$ . The morphism  $\tau_{\mathcal{S}}$  is well defined if for any differential polynomial  $P$  such that  $P(X) = 0$  we have  $\tau_{\mathcal{S}}(P(X)) = 0$ . A differential algebraic relation for the fundamental solution matrix  $X$  can be seen as a  $\mathcal{F}$ -linear form that annihilates on a construction  $\mathcal{N}_{\mathcal{F}}$  of linear differential algebra of  $\mathcal{M}_{\mathcal{F}}$ . Since the set of  $\mathcal{F}$ -linear forms that vanish on  $\mathcal{N}_{\mathcal{F}}$  is a  $q$ -difference submodule of  $\mathcal{N}_{\mathcal{F}}^*$ , it must be stabilized by  $\tau$ . It follows that  $\tau_{\mathcal{S}}(P(X)) = 0$  for any differential polynomial  $P$  such that  $P(X) = 0$ . One can check that the compatibility of the sequence  $\tau$  with the tensor product and the prolongation functor implies that  $\alpha(\mathcal{S})(\tau_{\mathcal{S}}) = \tau$ .

To conclude, we have proved that for any  $\partial$ - $\mathcal{C}$ -algebra, the  $\alpha(\mathcal{S})$ 's are isomorphisms. This proves that  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega_R)$  and  $G_R^\partial$  are isomorphic over  $\mathcal{C}$ .  $\square$

As in Remark 1.13, we consider a finitely generated extension  $K/\mathbb{Q}$ , an element  $q \in K \setminus \{0\}$  and a field embedding  $K \hookrightarrow \mathbb{C}$ , such that  $|q| \neq 1$  for the usual norm of  $\mathbb{C}$ . Let  $\mathcal{M}_{K(x)}$  be a  $q$ -difference module over  $K(x)$ .

**Notation 3.6.** For any  $q$ -difference field extension  $\mathcal{F}/K(x)$  we will denote by  $\mathcal{M}_{\mathcal{F}}$  the  $q$ -difference module over  $\mathcal{F}$  obtained from  $\mathcal{M}_{K(x)}$  by scalar extension. More precisely,  $\mathcal{M}_{\mathcal{F}} = M_{K(x)} \otimes_{K(x)} \mathcal{F}$  and  $\Sigma_q$  is defined on  $\mathcal{M}_{\mathcal{F}}$  by  $\Sigma_q \otimes \sigma_q$ .

Let  $\sigma_q(\vec{y}) = A\vec{y}$  be the  $q$ -difference system associated to  $\mathcal{M}_{K(x)}$  with respect to a fixed basis. We will consider as in §1.3 the weak Picard-Vessiot rings  $R$ ,  $R_E$  and  $\tilde{R}$ , extending conveniently the constants to  $\mathbb{C}$ ,  $\mathcal{C}_E$  and  $\tilde{\mathcal{C}}_E$  respectively. In parallel, following Proposition 3.4, each of these weak parameterized Picard-Vessiot rings yields to a neutral differential fiber functor for  $\langle \mathcal{M}_{\tilde{\mathcal{C}}_E(x)} \rangle^{\otimes, \partial}$ ,  $\langle \mathcal{M}_{\mathbb{C}(x)} \rangle^{\otimes, \partial}$ , and  $\langle \mathcal{M}_{\mathcal{C}_E(x)} \rangle^{\otimes, \partial}$ , respectively. When restricted to the Tannakian category  $\langle \mathcal{M}_{\tilde{\mathcal{C}}_E(x)} \rangle^{\otimes}$ ,  $\langle \mathcal{M}_{\mathbb{C}(x)} \rangle^{\otimes}$ , and  $\langle \mathcal{M}_{\mathcal{C}_E(x)} \rangle^{\otimes}$ , these differential fiber functors induce neutral fiber functors in the classical sense of [Del90]. Proposition 3.5 immediately implies the following:

**Corollary 3.7.** *We have:*

- $G_R^\partial \cong \text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}(x)}, \omega_R)$  over  $\mathbb{C}$ .
- $G_{R_E}^\partial \cong \text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E})$  over  $\mathbb{C}_E$ .
- $G_{\tilde{R}}^\partial \cong \text{Aut}^{\otimes, \partial}(\mathcal{M}_{\tilde{\mathbb{C}}_E(x)}, \omega_{\tilde{R}})$  over  $\tilde{\mathbb{C}}_E$ .

### 3.3 List of all fiber functors

For the reader convenience we remind the list of all neutral differential fiber functors defined above:

$$\begin{aligned} \omega_R &: \langle \mathcal{M}_{\mathbb{C}(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\mathbb{C}}, & \mathcal{N} &\mapsto \ker(\Sigma_q - Id, R \otimes_{\mathbb{C}(x)} \mathcal{N}), \\ \omega_{R_E} &: \langle \mathcal{M}_{\mathbb{C}_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\mathbb{C}_E}, & \mathcal{N} &\mapsto \ker(\Sigma_q - Id, R_E \otimes_{\mathbb{C}_E(x)} \mathcal{N}), \\ \omega_{\tilde{R}} &: \langle \mathcal{M}_{\tilde{\mathbb{C}}_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\tilde{\mathbb{C}}_E}, & \mathcal{N} &\mapsto \ker(\Sigma_q - Id, \tilde{R} \otimes_{\tilde{\mathbb{C}}_E(x)} \mathcal{N}), \end{aligned}$$

whose associated groups are  $G_R^\partial$ ,  $G_{R_E}^\partial$  and  $G_{\tilde{R}}^\partial$ . Moreover we have the four forgetful functors:

$$\begin{aligned} \eta_{K(x)} &: \langle \mathcal{M}_{K(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{K(x)}, \\ \eta_{\mathbb{C}(x)} &: \langle \mathcal{M}_{\mathbb{C}(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\mathbb{C}(x)}, \\ \eta_{\mathbb{C}_E(x)} &: \langle \mathcal{M}_{\mathbb{C}_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\mathbb{C}_E(x)}, \\ \eta_{\tilde{\mathbb{C}}_E(x)} &: \langle \mathcal{M}_{\tilde{\mathbb{C}}_E(x)} \rangle^{\otimes, \partial} \longrightarrow \text{Vect}_{\tilde{\mathbb{C}}_E}, \end{aligned}$$

that define the intrinsic Galois groups  $\text{Gal}^\partial(\mathcal{M}_{K(x)})$ ,  $\text{Gal}^\partial(\mathcal{M}_{\mathbb{C}(x)})$ ,  $\text{Gal}^\partial(\mathcal{M}_{\mathbb{C}_E(x)})$  and  $\text{Gal}^\partial(\mathcal{M}_{\tilde{\mathbb{C}}_E(x)})$ , respectively. We will call by the same name the restrictions of the functors above to the usual Tannakian categories  $\langle \mathcal{M}_{K(x)} \rangle^\otimes$ ,  $\langle \mathcal{M}_{\mathbb{C}(x)} \rangle^\otimes$ ,  $\langle \mathcal{M}_{\mathbb{C}_E(x)} \rangle^\otimes$ , and  $\langle \mathcal{M}_{\tilde{\mathbb{C}}_E(x)} \rangle^\otimes$ . Using Notation 3.1 for the groups, i.e., dropping the superscript  $\partial$ , we obtain the following Tannakian groups:  $\text{Gal}(\mathcal{M}_{K(x)})$ ,  $\text{Gal}(\mathcal{M}_{\mathbb{C}(x)})$ ,  $\text{Gal}(\mathcal{M}_{\mathbb{C}_E(x)})$ ,  $\text{Gal}(\mathcal{M}_{\tilde{\mathbb{C}}_E(x)})$ , respectively. One can consider the difference Galois groups  $G_R$ ,  $G_{R_E}$ ,  $G_{\tilde{R}}$ , defined in Remark 1.12. Notice that the analogue of Corollary 3.7 for  $G_R$ ,  $G_{R_E}$ ,  $G_{\tilde{R}}$  is well known (see [Del90, §9.4]).

## 4 Comparison theorems

One of the main results of [CHS08, §3] is (see also [PN11], for a model theoretic approach):

**Theorem 4.1.** *The group schemes  $G_R$ ,  $G_{R_E}$  and  $G_{\tilde{R}}$  become isomorphic over  $\tilde{\mathbb{C}}_E$ .*

*Remark 4.2.* In [Sau03], Sauloy constructs a  $\mathbb{C}$ -linear fiber functor for  $q$ -difference modules over  $\mathbb{C}(x)$ , using a basis of meromorphic solutions. Since  $\mathbb{C}$  is algebraically closed, it follows from the classical general theory of Tannakian categories, that such a fiber functor gives rise to a group that is isomorphic to the Picard-Vessiot group of [vdPS97] over  $\mathcal{F} = \mathbb{C}(x)$ . We won't consider Sauloy's point of view in this paper.

One of the most important properties of functional Galois groups is that their dimension as algebraic variety is equal to the transcendence degree of the associated Picard-Vessiot rings. In particular, the sets of the entries of any fundamental solution matrix of (1.1) in  $R$ ,  $R_E$  or  $\tilde{R}$  have the same transcendence degree over the associated base field, i.e.,  $\mathbb{C}(x)$ ,  $\mathbb{C}_E(x)$  or  $\tilde{\mathbb{C}}_E(x)$ , respectively.

We have the following differential analogue of the Theorem above:

**Theorem 4.3.** *In the previous notation,  $G_R^\partial \otimes_{\mathbb{C}} \tilde{\mathbb{C}}_E \simeq G_{R_E}^\partial \otimes_{\mathbb{C}_E} \tilde{\mathbb{C}}_E \simeq G_{\tilde{R}}^\partial$ .*

*Remark 4.4.* The proof below is a parameterized analog of [CHS08, Corollary 2.5].

*Proof.* Let  $S := \mathbb{C}_E(x)\{Y, \det Y^{-1}\}_{\partial}/q\mathbb{C}_E(x)$  be the PPV ring over  $\mathbb{C}_E(x)$  defined as in Corollary 1.22 and let  $\phi : R \otimes_{\mathbb{C}} \mathbb{C}_E \rightarrow S$  be the embedding considered in the proof of Proposition 1.21. The group  $G_R^{\partial}$  is a functor from  $\partial$ - $\mathbb{C}$ -algebras  $A$  to groups defined by  $G_R^{\partial}(A) = \text{Aut}_{\mathbb{C}(x) \otimes A}^{(\sigma_q, \partial)}(R \otimes_{\mathbb{C}} A)$ . We define analogously  $G_S^{\partial}$  as a functor from  $\partial$ - $\mathbb{C}_E$ -algebras to groups. By Proposition 2.8, these functors are representable. (See Appendix B.) Let  $T_R$  be the finitely generated  $\partial$ - $\mathbb{C}$ -algebra representing  $G_R^{\partial}$  and let  $T_S$  be the finitely generated  $\partial$ - $\mathbb{C}_E$ -algebra representing  $G_S^{\partial}$ . We define a new functor  $F$  from  $\partial$ - $\mathbb{C}_E$ -algebras  $B$  to groups as  $F(B) = \text{Aut}_{\mathbb{C}_E(x) \otimes B}^{(\sigma_q, \partial)}((R \otimes_{\mathbb{C}} \mathbb{C}_E) \otimes_{\mathbb{C}_E} B)$ . One can easily check that  $F$  is representable by  $T_R \otimes_{\mathbb{C}} \mathbb{C}_E$ . Using the embedding  $\phi$ , one sees that

$$(4.1) \quad F(B) = \text{Aut}_{\mathbb{C}_E(x) \otimes B}^{(\sigma_q, \partial)}(S \otimes_{\mathbb{C}_E} B) = G_R(B),$$

for any  $\partial$ - $\mathbb{C}$ -algebra  $B$ . Yoneda Lemma (see Appendix B) yields to  $T_R \otimes_{\mathbb{C}} \mathbb{C}_E \simeq T_S$ , which is  $G_R^{\partial} \otimes_{\mathbb{C}} \mathbb{C}_E \simeq G_S^{\partial}$ . A similar argument shows that the isomorphism of  $(\sigma_q, \partial)$ - $\widetilde{\mathbb{C}_E}$ -algebras between  $S \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E}$  and  $R_E \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E}$  yields to the isomorphism  $G_S^{\partial} \otimes_{\mathbb{C}} \widetilde{\mathbb{C}_E} \simeq G_{R_E}^{\partial} \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E}$ . This proves that  $G_R^{\partial} \otimes_{\mathbb{C}} \widetilde{\mathbb{C}_E} \simeq G_{R_E}^{\partial} \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E}$ .

Replacing  $S$  with  $\widetilde{S}$  (see Corollary 1.22), one shows in the same way that  $G_R^{\partial} \otimes_{\mathbb{C}} \widetilde{\mathbb{C}_E} \simeq G_{\widetilde{R}}^{\partial} \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E}$ .  $\square$

*Remark 4.5.* By [HS08, Prop. 6.21], the Zariski closure of  $G_R^{\partial}$ ,  $G_{R_E}^{\partial}$  and  $G_{\widetilde{R}}^{\partial}$  coincide with  $G_R$ ,  $G_{R_E}$  and  $G_{\widetilde{R}}$ , respectively. Therefore we can retrieve the Theorem 4.1 as a corollary of Theorem 4.3.

We are now concerned with the intrinsic Galois groups, both parameterized and not. Let  $\mathcal{M}_{K(x)}$  be a  $q$ -difference module defined over  $K(x)$ , with  $K$  a finite generated extension of  $\mathbb{Q}$ .

For a  $q$ -difference module  $\mathcal{M}_{\mathcal{F}}$  over  $\mathcal{F}$ , the comparison between the intrinsic Galois group and the group of tensor automorphism of a neutral fiber functor  $\omega$  for  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes}$  is a direct consequence of the fact that  $\text{Hom}^{\otimes}(\omega, \eta_{\mathcal{F}})$ , which is a bitorsor on  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega)$  and  $\text{Gal}(\mathcal{M}_{\mathcal{F}})$ , is also an  $\mathcal{F}$ -scheme and has therefore a point in some algebraically closed extension  $\widetilde{\mathcal{F}}$  of  $\mathcal{F}$ . This point gives rise to an isomorphism over  $\widetilde{\mathcal{F}}$  between  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathcal{F}}, \omega)$  and  $\text{Gal}(\mathcal{M}_{\mathcal{F}})$ . A similar result holds in the differential parameter context. More, precisely, [GGO13, Proposition 4.25] shows that, when  $\omega$  is a neutral differential fiber functor for  $\langle \mathcal{M}_{\mathcal{F}} \rangle^{\otimes, \partial}$ , the functor  $\text{Hom}^{\otimes, \partial}(\omega, \eta_{\mathcal{F}})$  is a  $\partial$ - $\mathcal{F}$ -scheme. As above, this yields an isomorphism between  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathcal{F}}, \omega)$  and  $\text{Gal}^{\partial}(\mathcal{M}_{\mathcal{F}})$  over a differentially closed field extension of  $\mathcal{F}$ . In our  $q$ -difference setting, this leads to the following statement:

**Proposition 4.6.** *Let us denote by  $\widetilde{\mathbb{C}(x)}$  (resp.  $\widetilde{\mathbb{C}_E(x)}$ ) a differential closure of  $\mathbb{C}(x)$  (resp.  $\mathbb{C}_E(x)$ ). We have the following isomorphisms of group-schemes:*

1.  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathbb{C}(x)}, \omega_R) \otimes_{\mathbb{C}} \widetilde{\mathbb{C}(x)} \simeq \text{Gal}(\mathcal{M}_{\mathbb{C}(x)}) \otimes_{\mathbb{C}(x)} \widetilde{\mathbb{C}(x)}$ ;
2.  $\text{Aut}^{\otimes}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E}) \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E(x)} \simeq \text{Gal}(\mathcal{M}_{\mathbb{C}_E(x)}) \otimes_{\mathbb{C}_E(x)} \widetilde{\mathbb{C}_E(x)}$ ;

and the following isomorphisms of  $\partial$ -group-schemes:

- 1bis.  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}(x)}, \omega_R) \otimes_{\mathbb{C}} \widetilde{\mathbb{C}(x)} \simeq \text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}(x)}) \otimes_{\mathbb{C}(x)} \widetilde{\mathbb{C}(x)}$ ;
- 2bis.  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E}) \otimes_{\mathbb{C}_E} \widetilde{\mathbb{C}_E(x)} \simeq \text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}_E(x)}) \otimes_{\mathbb{C}_E(x)} \widetilde{\mathbb{C}_E(x)}$ .

Since the dimension of a  $\partial$ -group-scheme as well as the differential transcendence degree (see Definition A.3) of a field extension is stable up to field extension, one obtains the following corollary:

**Corollary 4.7.** *Let  $\mathcal{M}_{K(x)}$  be a  $q$ -difference module defined over  $K(x)$ . Let  $U \in \text{GL}(\mathcal{M}(\mathbb{C}^*))$  be a fundamental solution matrix attached to  $\mathcal{M}_{K(x)}$ , as in Proposition 1.14.*

*Then, the differential transcendence degree of the differential field  $F_E$  generated over  $\mathbb{C}_E(x)$  by the entries of  $U$  is equal to the differential dimension of  $\text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$  over  $\mathbb{C}(x)$ .*

*Proof.* By [GGO13, Proposition 4.28], the functor  $\text{Isom}^{\otimes, \partial}(\omega_{R_E} \otimes_{\mathbb{C}_E(x)}, \eta_{\mathbb{C}_E(x)})$  is a reduced  $\partial$ - $\mathbb{C}_E(x)$ -scheme, represented by  $R_E$ . It is also a  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E})$ -torsor. It has thus a  $\widetilde{\mathbb{C}_E(x)}$ -point, which gives, by triviality of the torsor, a  $(\sigma_q, \partial)$ -isomorphism between  $\widetilde{\mathbb{C}_E(x)} \otimes_{\mathbb{C}_E(x)} R_E$  and  $\widetilde{\mathbb{C}_E(x)} \otimes_{\mathbb{C}_E}$

$\mathbb{C}_E\{\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E})\}$ . Using the discussion on the differential dimension in Appendix B, we get that the differential dimension of  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E})$  equals the differential transcendence degree of  $F_E$  over  $\mathbb{C}_E(x)$ . By Proposition 4.6 combined with Proposition 4.3, we find that  $\text{Aut}^{\otimes, \partial}(\mathcal{M}_{\mathbb{C}_E(x)}, \omega_{R_E})$  is isomorphic to  $\text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$  over  $\tilde{\mathbb{C}}_E(x)$ . We conclude by using one more time the fact that the differential dimension of a reduced  $\partial$ -scheme is invariant by base field extension.  $\square$

In [Kat87, Lemma 1.3.2], it is shown that the group of tensor automorphisms of a  $K$ -linear neutral fiber functor is invariant up to algebraic field extension of  $K$ . For forgetful functors, this is not true. This is essentially due to the fact that, unlike to the case of neutral fiber functors, a vector space stable under the action of the group of tensor automorphisms of the forgetful functor is not necessarily an object of the Tannakian category. However, one can show that, for any field extension  $L/K$ , the parameterized intrinsic Galois group of  $\mathcal{M}_{L(x)}$  is equal, up to scalar extension, to the parameterized intrinsic Galois group of  $\mathcal{M}_{K'(x)}$ , for a suitable finitely generated extension  $K'/K$ , with  $K' \subset L$ .<sup>1</sup>

**Lemma 4.8.** *Let  $L$  be a field extension of  $K$  with  $\sigma_q|_L = \text{id}$ . There exists a finitely generated intermediate field  $L/K'/K$  such that*

$$\text{Gal}(\mathcal{M}_{L(x)}) \cong \text{Gal}(\mathcal{M}_{K'(x)}) \otimes_{K'(x)} L(x)$$

and

$$\text{Gal}^{\partial}(\mathcal{M}_{L(x)}) \cong \text{Gal}^{\partial}(\mathcal{M}_{K'(x)}) \otimes_{K'} L(x).$$

These equalities hold when we replace  $K'$  by any subfield extension of  $L$  containing  $K'$ .

*Proof.* By definition,  $\text{Gal}^{\partial}(\mathcal{M}_{L(x)})$  is the stabilizer inside  $\text{GL}(M_{L(x)})$  of all  $L(x)$ -vector spaces of the form  $W_{L(x)}$  for  $W$  object of  $\langle \mathcal{M}_{L(x)} \rangle^{\otimes, \partial}$ .

Similarly, for any field extension  $L/K'/K$ , we have an equality

$$\text{Gal}^{\partial}(\mathcal{M}_{K'(x)}) = \text{Stab}(W_{K'(x)}, W \text{ object of } \langle \mathcal{M}_{K'(x)} \rangle^{\otimes, \partial}),$$

that has to be understood as a functorial equality for differential scheme defined above  $L(x)$ . Then,

$$\text{Gal}^{\partial}(\mathcal{M}_{L(x)}) \subset \text{Gal}^{\partial}(\mathcal{M}_{K'(x)}) \otimes L(x).$$

By noetherianity, the (parameterized) intrinsic Galois group of  $\mathcal{M}_{L(x)}$  is defined by a finite family of (differential) polynomial equations, thus we can choose  $K'$ , which contains the coefficients of the defining equations.  $\square$

The corollary below summarizes results of this chapter.

**Corollary 4.9.** *Let  $\mathcal{M}_{K(x)}$  be a  $q$ -difference module defined over  $K(x)$ . Let  $U \in \text{GL}_{\nu}(\mathcal{M}(C^*))$  be a fundamental matrix of meromorphic solutions of  $\mathcal{M}_{K(x)}$ . Then,*

1. *the dimension of  $\text{Gal}(\mathcal{M}_{\mathbb{C}(x)})$  is equal to the transcendence degree of the field generated by the entries of  $U$  over  $\mathbb{C}_E(x)$ , i.e., the  $\mathbb{C}(x)$ -group-scheme  $\text{Gal}(\mathcal{M}_{\mathbb{C}(x)})$  measures the algebraic relations between the meromorphic solutions of  $\mathcal{M}_{\mathbb{C}_E(x)}$ .*
2. *the differential dimension of  $\text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$  is equal to the differential transcendence degree of the differential field generated by the entries of  $U$  over  $\tilde{\mathbb{C}}_E(x)$ , i.e., the  $\partial$ - $\mathbb{C}(x)$ -group-scheme  $\text{Gal}^{\partial}(\mathcal{M}_{\mathbb{C}(x)})$  encodes the differential algebraic relations between the meromorphic solutions of  $\mathcal{M}_{K(x)}$ .*
3. *there exists a finitely generated extension  $K'/K$  such that the differential transcendence degree of the differential field generated by the entries of  $U$  over  $\tilde{\mathbb{C}}_E(x)$  is equal to the differential dimension of  $\text{Gal}^{\partial}(\mathcal{M}_{K'(x)})$ , i.e., it is given by an arithmetic characterization (see Theorem 3.3).*

*Proof.* The first two statements are proved in Corollary 4.7. The third one is Lemma 4.8.  $\square$

<sup>1</sup>In [BCDVW19], for differential modules, the authors optimize the field on which such an isomorphism is true, using an effective characterization of Kolchin's reduced forms.

## Appendix

We quickly recall some basic facts of differential algebra as well as some very basic notions of differential algebraic geometry, mainly in the affine case. We refer to [Kol73] and [Kov02] for a detailed exposition.

### A Differential algebra

We largely use standard notation of differential algebra as can be found in [Kol73]. A differential ring (or  $\partial$ -ring for short) is a ring  $R$  together with a derivation  $\partial : R \rightarrow R$ , i.e., an additive map  $\partial : R \rightarrow R$  satisfying the Leibniz rule  $\partial(ab) = \partial(a)b + a\partial(b)$ , for all  $(a, b) \in R^2$ . The ring of  $\partial$ -constants of  $R$  is  $R^\partial = \{r \in R \mid \partial(r) = 0\}$ . All rings considered in this work are commutative with identity and all *differential* rings contain the ring of integer numbers. In particular, all fields are of characteristic zero.

Given two  $\partial$ -rings  $(R, \partial)$  and  $(R', \partial')$ , a morphism  $\psi : R \rightarrow R'$  of  $\partial$ -rings is a morphism of rings such that  $\psi\partial = \partial'\psi$ .

A  $\partial$ -ideal  $\mathcal{I}$  of a  $\partial$ -ring  $R$  is an ideal of  $R$  that is invariant under the action of  $\partial$ . A  $\partial$ -ring  $R$  is said to be  $\partial$ -simple if it does not contain any non-zero proper  $\partial$ -ideals.

A  $\partial$ -field  $k$  is a field that is also a  $\partial$ -ring. A  $\partial$ - $k$ -algebra  $R$  is a  $k$ -algebra and a  $\partial$ -ring such that the morphism  $k \rightarrow R$  is a morphism of  $\partial$ -rings. Given two  $\partial$ - $k$ -algebras  $(R, \partial)$  and  $(R', \partial')$ , a morphism  $\psi : R \rightarrow R'$  of  $\partial$ - $k$ -algebras is a morphism of  $k$ -algebras such that  $\psi\partial = \partial'\psi$ . If, moreover,  $R$  is a  $\partial$ -field and a  $\partial$ - $k$ -algebra, we say that  $R|k$  is a  $\partial$ -field extension.

Let  $k$  be a  $\partial$ -field and  $R$  a  $\partial$ - $k$ -algebra. If  $B$  is a subset of  $R$ , then  $k\langle B \rangle_\partial$  denotes the smallest  $\partial$ - $k$ -subalgebra of  $R$  that contains  $B$ . If  $R = k\langle B \rangle_\partial$  for some finite subset  $B$  of  $R$ , we say that  $R$  is finitely  $\partial$ -generated over  $k$ . If  $K|k$  is an extension of  $\partial$ -fields and  $B \subset K$ , then  $k\langle B \rangle_\partial$  denotes the smallest  $\partial$ -field extension of  $k$  inside  $K$  that contains  $B$ .

**Definition A.1.** The  $\partial$ - $k$ -algebra  $k\{x\}_\partial = k\{x_1, \dots, x_n\}_\partial$  of  $\partial$ -polynomials over  $k$  in the  $\partial$ -variables  $x_1, \dots, x_n$  is the polynomial ring over  $k$  in the countable set of algebraically independent variables  $x_1, \dots, x_n, \partial(x_1), \dots, \partial(x_n), \dots$ , with an action of  $\partial$  as suggested by the names of the variables.

Of course, for any  $\partial$ -field extension  $L|k$  and any  $f := (f_1, \dots, f_n) \in L^n$ , one has a  $\partial$ - $k$ -morphism from  $k\{x\}_\partial$  to  $L$ , which assigns  $x_i$  to  $f_i$ , for all  $i = 1, \dots, n$ . We say that  $f$  is a solution of the differential algebraic equation  $P(x) = 0$ , for some  $P \in k\{x\}_\partial$ , if  $P$  lies in the kernel of the specialization morphism above.

**Definition A.2.** A  $\partial$ -field  $k$  is called differentially closed or  $\partial$ -closed, for short, if any system of differential algebraic equations with coefficients in  $k$ , having a solution in some differential field extension of  $k$ , has a solution in  $k$ . A differential closure of a  $\partial$ -field  $k$  is a  $\partial$ -field extension of  $k$  that is  $\partial$ -closed and that embeds, as  $\partial$ -field extension of  $k$ , in any differentially closed extension of  $k$ .

**Definition A.3.** Let  $L|K$  be a  $\partial$ -field extension. Elements  $a_1, \dots, a_n \in L$  are called differentially (or  $\partial$ -algebraically) independent over  $K$  if the elements  $a_1, \dots, a_n, \partial(a_1), \dots, \partial(a_n), \dots$  are algebraically independent over  $K$ . Otherwise, they are called differentially dependent over  $K$ .

A  $\partial$ -transcendence basis of  $L$  over  $K$  is a subset of  $L$ , which is maximal with respect to the property of being a differentially independent set over  $K$ .

Any two  $\partial$ -transcendence basis of  $L|K$  have the same cardinality and so we can define the  $\partial$ -transcendence degree of  $L|K$  (or differential transcendence degree of  $L|K$ , when the choice of  $\partial$  is clear, or also  $\partial$ -trdeg( $L|K$ ), for short) as the cardinality of any  $\partial$ -transcendence basis of  $L$  over  $K$ .

### B Differential geometry

In this paper, we work with the formalism of affine differential group schemes, as can be found in [Kov02]. In this section, we fix a  $\partial$ -field  $k$  of characteristic zero, not necessarily  $\partial$ -closed. We define a  $\partial$ - $k$ -scheme as follows:

**Definition B.1.** An affine  $\partial$ -scheme over  $k$  (or an affine  $\partial$ - $k$ -scheme, for short) is a (covariant) functor from the category of  $\partial$ - $k$ -algebras to the category of sets which is representable.

The definition above means that a functor  $X$  from the category of  $\partial$ - $k$ -algebras to the category of sets is a  $\partial$ - $k$ -scheme if and only if there exists a  $\partial$ - $k$ -algebra  $k\{X\}$  and an isomorphism of functors  $X \simeq \text{Alg}_k^\partial(k\{X\}, -)$ , where  $\text{Alg}_k^\partial$  stands for morphism of  $\partial$ - $k$ -algebras. By the Yoneda lemma, the  $\partial$ - $k$ -algebra  $k\{X\}$  is uniquely determined up to unique  $\partial$ - $k$ -isomorphisms. We call it the ring of  $\partial$ -coordinates of  $X$ . A  $\partial$ - $k$ -scheme  $X$  is called  $\partial$ -algebraic (over  $k$ ) if  $k\{X\}$  is finitely  $\partial$ -generated over  $k$ . We say that a  $\partial$ - $k$ -scheme  $X$  is reduced if  $k\{X\}$  has no non-zero nilpotent elements.

Let  $X$  be a  $\partial$ - $k$ -scheme. By a closed  $\partial$ - $k$ -subscheme  $Y \subset X$  we mean a subfunctor  $Y$  of  $X$  which is represented by  $k\{X\}/\mathbb{I}(Y)$  for some  $\partial$ -ideal  $\mathbb{I}(Y)$  of  $k\{X\}$ . The ideal  $\mathbb{I}(Y)$  of  $k\{X\}$  is uniquely determined by  $Y$  and vice versa. We call it the vanishing ideal of  $Y$  in  $X$ .

A morphism of  $\partial$ - $k$ -schemes is a morphism of functors. If  $\phi: X \rightarrow Y$  is a morphism of  $\partial$ - $k$ -schemes, we denote the dual morphism of  $\partial$ - $k$ -algebras with  $\phi^*: k\{Y\} \rightarrow k\{X\}$ .

If a functor (resp.  $\partial$ -functor)  $X$  factors through the category of group, we say that  $X$  is a  $k$ -group-scheme (resp.  $\partial$ - $k$ -group-scheme). We denote by  $\text{GL}_\nu(k)$  the  $\partial$ - $k$ -groupscheme attached to the general linear group of size  $\nu$  over  $k$ . It is represented by the  $\partial$ - $k$ -algebra  $k\{X, \det X^{-1}\}_\partial$  where  $X$  is a  $\nu \times \nu$  matrix of  $\partial$ -indeterminates. More generally, for any  $k$ -vector space  $V$  of finite dimension, we denote by  $\text{GL}(V)$  the  $\partial$ - $k$ -groupscheme of invertible  $k$ -linear automorphisms of  $V$ . Notice that we are calling  $\text{GL}_\nu(k)$  both the  $k$ -groupscheme and the  $\partial$ - $k$ -groupscheme attached to the general linear group, anyway the context will always make clear to which one of the two structures we are referring to, without introducing complicate notation.

By a  $\partial$ -subgroup  $H$  of  $G$ , we mean a  $\partial$ - $k$ -scheme  $H$  such that  $H(S)$  is a subgroup of  $G(S)$  for every  $\partial$ - $k$ -algebra  $S$ . We call  $H$  normal if  $H(S)$  is a normal subgroup of  $G(S)$  for every  $\partial$ - $k$ -algebra  $S$ . As in the classical setting, Yoneda lemma implies that, for a  $\partial$ - $k$ -group-scheme  $G$ , the algebra  $k\{G\}$  is a  $\partial$ - $k$ -Hopf algebra, i.e., a  $\partial$ - $k$ -algebra equipped with the structure of a Hopf algebra over  $k$  such that the Hopf algebra structure maps are morphisms of  $\partial$ -rings. It also follows immediately that the category of  $\partial$ - $k$ -group-schemes is anti-equivalent to the category of  $\partial$ - $k$ -Hopf algebras. Then, since Hopf algebras over fields of characteristic zero are reduced by [Wat79, Cartier's Theorem in §11.4], we get that any  $\partial$ - $k$ -group-scheme is automatically reduced. Reduced  $\partial$ -schemes correspond to differential varieties in the sense of Kolchin (see for instance [Kol73]), for whom it suffices to focus on the solution set of a system of differential equations with value in a sufficiently big field, i.e., a  $\partial$ -closed field.

The  $\partial$ -schemes considered in this paper are all reduced. Thus, we only define the differential dimension of a reduced  $\partial$ -scheme. So let  $V$  be a reduced  $\partial$ - $k$ -scheme. We can write  $k\{V\} = k\{x_1, \dots, x_n\}_\partial/\mathfrak{q}$  for some positive integer  $n$  and some radical  $\partial$ -ideal  $\mathfrak{q} \subset k\{x_1, \dots, x_n\}_\partial$ . Since  $\mathfrak{q}$  is radical, by [Kap57, Theorem 7.5] there exists finitely many prime  $\partial$ -ideals  $\mathfrak{p}_i$  such that  $\mathfrak{q} = \cap \mathfrak{p}_i$ . Now, we can define the differential dimension of  $V$  over  $k$ , denoted by  $\partial\text{-dim}(V|k)$  as the maximum of the  $\partial$ -trdeg( $L_i|k$ ) where  $L_i$  denotes the fraction field of  $k\{x_1, \dots, x_n\}_\partial/\mathfrak{p}_i$ . In [Kol73, III.§6.Proposition 3], Kolchin proved that if  $k \subset k'$  is an extension of  $\partial$ -field and if  $V$  is a reduced  $\partial$ - $k$ -scheme, then  $\partial\text{-dim}(V|k) = \partial\text{-dim}(V_{k'}|k')$ , where  $V_{k'}$  is the base extension of  $V$  to  $k'$ .

Let  $V$  be a  $k$ -scheme, i.e., a (covariant) functor from the category of  $k$ -algebras to the category of sets which is representable by a  $k$ -algebra  $k[V]$ . We call  $k[V]$  the ring of coordinates of  $V$ . In [Gil02], the author shows that the forgetful functor

$$\eta : \partial\text{-}k\text{-algebras} \rightarrow k\text{-algebras},$$

that associates to any  $\partial$ - $k$ -algebra its underlying  $k$ -algebra, has a left adjoint denoted by  $D$ . This implies that the functor  $\mathbf{V}$  from the category of  $\partial$ - $k$ -algebras to the category of Sets, defined by the composition of  $V$  with the forgetful functor  $\eta$  is a  $\partial$ - $k$ -scheme, whose ring of  $\partial$ -coordinates is precisely  $D(k[V])$ . We call  $\mathbf{V}$ , the  $\partial$ - $k$ -scheme attached to  $V$ . The simple idea behind this construction is that polynomial equations are  $\partial$ -polynomials. More precisely if  $V \subset \mathbf{A}_k^n$ , the affine space of dimension  $n$  over  $k$ , and if  $I(V) \subset k[x_1, \dots, x_n]$  is the vanishing ideal of  $V$  as subscheme of  $\mathbf{A}_k^n$  then the vanishing ideal of  $\mathbf{V}$  as  $\partial$ - $k$ -subscheme of  $\mathbf{A}_k^n$  is nothing else than the  $\partial$ -ideal generated by  $I(V)$  in  $k\{x_1, \dots, x_n\}_\partial$ . Finally, Kolchin irreducibility theorem states that if  $k[V]$  is a finitely generated integral  $k$ -algebra, then  $D(k[V])$  is a finitely  $\partial$ -generated integral  $\partial$ - $k$ -algebra and the dimension of  $V$  as  $k$ -scheme coincides with the  $\partial$ -dimension of  $\mathbf{V}$  over  $k$  ([Gil02, §2]).

Conversely, given a  $\partial$ - $k$ -subscheme  $\mathbf{V}$  of some  $\mathbf{A}_k^n$ , we can attach to  $\mathbf{V}$  a  $k$ -subscheme of  $\mathbf{A}_k^n$  as follows. Let  $\mathbb{I}(\mathbf{V}) \subset k\{x_1, \dots, x_n\}_\partial$  be the vanishing ideal of  $\mathbf{V}$  in  $\mathbf{A}_k^n$ . Let  $\mathbf{V}^Z$  be the  $k$ -subscheme of  $\mathbf{A}_k^n$  defined

by the ideal  $\mathbb{I}(V) \cap k[x_1, \dots, x_n]$ . We call  $\mathbf{V}^Z$  the Zariski closure of  $\mathbf{V}$  inside  $\mathbf{A}_k^n$ . If  $k$  is  $\partial$ -closed then  $\mathbf{V}^Z$  is the closure of  $\mathbf{V}$  with respect to the Zariski topology.

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