ARITHMETIC THEORY OF q-DIFFERENCE EQUATIONS

(G_q-FUNCTIONS AND q-DIFFERENCE MODULES OF TYPE G, GLOBAL q-GEVREY SERIES)

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Abstract. In the first part of the paper we give a definition of G_q-function and we establish a regularity result, obtained as a combination of a q-analogue of the André-Chudnovsky Theorem [And89, VI] and Katz Theorem [Kat70, §13]. In the second part of the paper, we combine it with some formal q-analogous Fourier transformations, obtaining a statement on the irrationality of special values of the formal q-Borel transformation of a G_q-function.

Contents

1. Introduction

A G-function, notion introduced by C.L. Siegel in 1929, is a formal power series $y = \sum_{n\geq 0} y_n x^n$ with coefficients in the field of algebraic numbers $\overline{\mathbb{Q}}$, such that:

(1) the series y is solution of a linear differential equation with coefficients in $\overline{\mathbb{Q}}(x)$ (condition that actually ensures that the coefficients of y are contained in a number field K);
(2) there exist a sequence of positive numbers $N_n \in \mathbb{N}$ and a positive constant C such that $N_n y_s$ is an integer of K for any $0 \leq s \leq n$ and $N_n \leq C^n$;
(3) for any immersion $K \hookrightarrow \mathbb{C}$, the image of y in $\mathbb{C}[[x]]$ is a convergent power series for the usual norm.

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Roughly speaking, a G-module is a, *a posteriori* fuchsien, $K(x)/K$-differential module whose (uniform part of) solutions are G-functions (*cf.* [Bom81], [CC85], [And89], [DGS94]). More formally, if $Y'(x) = G(x)Y(x)$ is the differential system associate with such a connection in a given basis, one can iterate it obtaining a family of the higher order differential systems $\frac{1}{n!}\frac{d^nY}{dx^n}(x) = G_{[n]}(x)Y(x)$. Our differential module is of type G if there exist a constant $C > 0$ and a sequence of polynomials $P_n(x) \in \mathbb{Z}[x]$, such that

(1) $P_n(x)G_{[s]}(x)$ is a matrix whose entries are polynomials with coefficients in the ring of integers of K, for any $s = 1, \dots, n$;
(2) the absolute value of the coefficients of $P_n(x)$ is smaller that C^n.

The unsolved Bombieri-Dwork's conjecture says that G-modules *come from geometry*, in the sense that they are extensions of direct summands of Gauss-Manin connections: the precise conjecture is stated in [And89, II]. Y. André proves that a differential module coming from geometry is of type G (*cf.* [And89, V, App.]). More recently, the theory of G-functions has been the starting point for the papers [And00a] and [And00b], where the author develops an arithmetic theory of Gevrey series, allowing for a new approach to some diophantine results, such as the Schidlovskii's theorem.

The question of the existence of an arithmetic theory of q-difference equations was first asked in [And00b]. A naive analogue over a number field of the notion above clearly does not work. In fact, let K be a number field and let $q \in K$, $q \neq 0$, not be a root of unity. We consider formal power series $y \in K[[x]]$ that satisfies conditions 2 and 3 of the definition of G-function given above and that is solution of a nontrivial q-difference equation with coefficients in $K(x)$, *i.e.* :

$$a_\nu(x)y(q^\nu x) + a_{\nu-1}y(q^{nu-1}x) + \cdots + a_0(x)y(x) = 0\,,$$

with $a_0(x), \dots, a_\nu(x) \in K(x)$, not all zero. Then the following result by Y. André is the key point of [DV02]:

Proposition 1.1 ([DV02, 8.4.1])**.** *A series y as above is the Taylor expansion at 0 of a rational function in $K(x)$.*

Other unsuccessful suggestions for a q-analogue of a G-function are made in [DV00, App.]. These considerations may induce to conclude that q-difference equations do not come from geometry over $\overline{\mathbb{Q}}$.

Here we propose another approach: we consider a finite extension K of the field of rational function $k(q)$ in q with coefficients in a field k. This is a very natural approach since in the literature, q very often considered as a parameter. Since K is a global field, we can define a G_q-function to be a series in $K[[x]]$, solution of a q-difference equation with coefficients in $K(x)$, satisfying a straightforward analogue of conditions 2 and 3 of the definition above. As far as the definition of q-difference modules of type G is concerned only the places of K modulo whom q is a root of unity - that we will briefly call cyclotomic places - comes into the picture (*cf.* Proposition 3.1 below). In fact, consider a q-difference system

$$Y(qx) = A(x)Y(x)\,, \tag{1.1.1}$$

with $A(x) \in Gl_\nu(K(x))$: its solutions can be interpreted as the horizontal vectors of a $K(x)$-free module M of rank ν with respect to a semilinear bijective operator Σ_q verifying $\Sigma_q(f(x)m) = f(qx)\Sigma_q(m)$ for any $f(x) \in K(x)$ and any $m \in M$. We consider the q-derivation:

$$d_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}$$

and its iterations:

$$\frac{d_q^n}{[n]_q^!}\,,\ \text{with } [0]_q^! = [1]_q^! = 1 \text{ and } [n]_q^! = \tfrac{q^n-1}{q-1}[n-1]_q^!.$$

We can obtain from (1.1.1) a whole family of systems:

$$\frac{d_q^n}{[n]_q^!}Y(x) = G_{[n]}(x)Y(x)\,,$$

where $G_1(x) = \frac{A(x)-1}{(q-1)x}$ and $\frac{q^n-1}{q-1}G_{[n]}(x) = G_{[1]}(x)G_{[n-1]}(qx) + d_qG_{[n-1]}(x)$. The fact that the denominators $[n]_q^!$ of the iterated derivations $\frac{d_q^n}{[n]_q^!}$ have positive valuation only at the cyclotomic places has the consequences that “there is no arithmetic growth” at the noncyclotomic places (*cf.* §3 below for a precise formulation). Moreover, an important role in the proofs is played by the reduction of q-difference systems modulo a cyclotomic place: this means that we specializes q to a root of unity and we study the nilpotence properties of the obtained system. In characteristic zero, one automatically obtain an iterative q-difference module, in the sense of C. Hardouin [Har07].

The role played by the cyclotomic valuations, and therefore by roots of unity, points out some analogies with other topics:

- The Volume Conjecture predicts a link between the hyperbolic volume of the complement of an hyperbolic knot and the asymptotic of the sequence $J_n(\exp(2i\pi/n))$, where $J_n(q)$ is an invariant of the knot called n-th Jones polynomial. The Jones polynomials are Laurent polynomials in q such that the generating series $\sum_{n\geq 0} J_n(q)x^n$ is solution of a q-difference equations with coefficients in $\mathbb{Q}(q)(x)$ (*cf.* [GL05]): the situation is quite similar to the one considered in the present paper. The q-difference equations appearing in this topological setting have, in general, irregular singularities, differently from the q-difference operators of type G, that are regular singular. To involve some irregular singular operators in the present framework, one should consider some formal q-Fourier transformations and develop a global theory of q-Gevrey series, in the wake of [And00a]: this is the topic of the second part of the paper.
- As already point out, an important role is played by the reduction of q-difference systems modulo the cyclotomic valuations. Conjecturally, the growth at cyclotomic places should be enough to describe the whole theory (*cf.* §3). It is natural to ask whether q-difference equations, that seem not to "come from geometry over $\overline{\mathbb{Q}}$", may have some geometric origin, in the sense of the geometry over $\mathbb{F}_1$ (*cf.* [Sou04], [CC08]).

Notice that in [Man08], Y. Manin establish a link between the Habiro ring, which is a topological algebra constructed to deal with quantum invariants of knots, and geometry over $\mathbb{F}_1$, so that the two remarks above are not orthogonal.

* * *

In the present paper we give a definition of G_q-functions and q-difference modules of type G. We test those definitions proving that a q-difference module having an injective solution whose entries are G-functions is of type G: that is to say that "the minimal q-difference module generated by a G-function" is of type G (*cf.* Theorem 4.2 below). We also prove that q-difference module of type G are regular singular (*cf.* Theorem 4.1). These two results are the base for the development of a global theory of q-Gevrey series.

In part two, we define global q-Gevrey series. Via the study of two q-analogues the formal Fourier transformation, we establish some structure theorems for the minimal q-difference equations killing global q-Gevrey series (*cf.* Theorems 12.3, 12.4 and 12.6). We conclude with an irrationality theorem for special values of of global q-Gevrey series of negative orders (*cf.* Theorem 13.6).

This paper won't be submitted for publication since the results below can be obtained in a more direct way. Namely, one can prove that G_q-functions are all rational (*cf.* [DVH09]). Nevertheless, the construction of the coefficients of the q-difference module from an injective solution in the proof of Theorem 4.2 has an interest in itself, since it may be applied to other difference operators.

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Part 1. G_q-FUNCTIONS AND q-DIFFERENCE MODULES OF TYPE G

2. DEFINITION AND FIRST PROPERTIES

Let us consider the field of rational function $k(q)$ with coefficients in a fixed field k. We fix $d \in (0,1)$ and for any irreducible polynomial $v = v(q) \in k[q]$ we set:

$$|f(q)|_v = d^{\deg_q v(q) \cdot \mathrm{ord}_{v(q)} f(q)}\,,\ \forall f(q) \in k[q]\,.$$

The definition of $|\ |_v$ extends to $k(q)$ by multiplicativity. To this set of norms one has to add the q^{-1}-adic one, defined on $k[q]$ by:

$$|f(q)|_{q^{-1}} = d^{-deg_q f(q)}\,;$$

once again this definition extends by multiplicativity to $k(q)$. Then the Product Formula holds:

$$\prod_v \left|\frac{f(q)}{g(q)}\right|_v = d^{\sum_v \deg_q v(q)\ \left(\mathrm{ord}_{v(q)} f(q) - \mathrm{ord}_{v(q)} g(q)\right)} = d^{deg_q f(q) - \deg_q g(q)} = \left|\frac{f(q)}{g(q)}\right|_{q^{-1}}^{-1} .$$

For any finite extension K of $k(q)$, we consider the family $\mathcal{P}$ of ultrametric norms, that extends the norms defined above, up to equivalence. We suppose that the norms in $\mathcal{P}$ are normalized so that the Product Formula still holds. We consider the following partition of $\mathcal{P}$:

- the set $\mathcal{P}_\infty$ of places of K such that the associated norms extend, up to equivalence, either $|\ |_q$ or $|\ |_{q^{-1}}$;
- the set $\mathcal{P}_f$ of places of K such that the associated norms extend, up to equivalence, one of the norms $|\ |_v$ for an irreducible $v = v(q) \in k[q]$, $v(q) \neq q$.

Moreover we consider the set $\mathcal{C}$ of places $v \in \mathcal{P}_f$ such that v divides a valuation of $k(q)$ having as uniformizer a factor of a cyclotomic polynomial. We will briefly call $v \in \mathcal{C}$ a cyclotomic place.

Definition 2.1. A series $y = \sum_{n\geq 0} y_n x^n \in K[[x]]$ is a *G_q-function* if:

(1) It is solution of a q-difference equations with coefficients in $K(x)$, *i.e.* there exists $a_0(x), \dots, a_\nu(x) \in K(x)$ not all zero such that

$$a_0(x)y(x) + a_1(x)y(qx) + \cdots + a_\nu(x)y(q^\nu x) = 0 . \tag{2.1.1}$$

(2) The series y has finite size, *i.e.*

$$\sigma(y) := \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\mathcal{P}} \log^+ \left(\sup_{s\leq n} |y_s|_v\right) < \infty ,$$

where $\log^+ x = \sup(0, \log x)$.

We will refer to the invariant σ as the *size*, using the same terminology as in the classical case of series over a number field.

Remark 2.2. (1) One can show that this definition of G_q-function is equivalent to the one given in the introduction (*cf.* [And89, I, 1.3]).

(2) Let $\overline{k(q)}$ be the algebraic closure of $k(q)$. A formal power series with coefficients in $\overline{k(q)}$ solution of a q-difference equations with coefficients in $\overline{k(q)}(x)$ is necessarily defined over a finite extension $K/k(q)$.

Proposition 2.3. *The set of G_q-functions is stable with respect to the sum and the Cauchy product*[1]*. Moreover, it is independent of the choice of K, in the sense that we can replace K by any finite extension of K.*

Proof. The proof is the same as in the case of classical G-functions (*cf.* [And89, I, 1.4, Lemma 2]). □

The field $K(x)$ is naturally a q-difference algebra, *i.e.* is equipped with the operator

$$\begin{array}{cccc} \sigma_q : & K(x) & \longrightarrow & K(x) \\ & f(x) & \longmapsto & f(qx) \end{array} .$$

The field $K(x)$ is also equipped with the q-derivation

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x} ,$$

satisfying a q-Leibniz formula:

$$d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x) ,$$

for any $f, g \in K(x)$. A q-difference module over $K(x)$ (of rank ν) is a finite dimensional $K(x)$-vector space M (of dimension ν) equipped with an invertible σ_q-semilinear operator, *i.e.*

$$\Sigma_q(f(x)m) = f(qx)\Sigma_q(m) ,$$

[1] It may be interesting to remark, although we won't need it in the sequel, that the estimate of the size of a product of G-functions proved in [And89, I, 1.4, Lemma 2] holds also in the case of G_q-functions.

for any $f \in K(x)$ and $m \in M$. A morphism of q-difference modules over $K(x)$ is a morphisms of $K(x)$-vector spaces, commuting to the q-difference structure (for more generalities on the topic, *cf.* [vdPS97], [DV02, Part I] or [DVRSZ03]).

Let $\mathcal{M} = (M, \Sigma_q)$ be a q-difference module over $K(x)$ of rank ν. We fix a basis $\underline{e}$ of M over $K(x)$ and we set:

$$\Sigma_q \underline{e} = \underline{e} A(x)\,,$$

with $A(x) \in Gl_\nu(K(x))$. An horizontal vector $\vec{y} \in K(x)^\nu$ with respect to Σ_q is a vector that verifies $\vec{y}(x) = A(x)\vec{y}(qx)$. Therefore we call

$$Y(qx) = A_1(x)Y(x)\,, \text{ with } A_1(x) = A(x)^{-1}\,,$$

the system associated to $\mathcal{M}$ with respect to the basis $\underline{e}$. Recursively we obtain the families of q-difference systems:

$$Y(q^n x) = A_n(x)Y(x) \text{ and } d_q^n Y(x) = G_n(x)Y(x)\,,$$

with $A_n(x) \in Gl_\nu(K(x))$ and $G_n(x) \in M_\nu(K(x))$. Notice that:

$$A_{n+1}(x) = A_n(qx)A_1(x)\,,\ G_1(x) = \frac{A_1(x) - 1}{(q-1)x} \text{ and } G_{n+1}(x) = G_n(qx)G_1(x) + d_q G_n(x)\,.$$

It is convenient to set $A_0 = G_0 = 1$. Moreover we set $[n]_q = \frac{q^n - 1}{q-1}$ for any $n \geq 1$, $[n]_q^! = [n]_q[n-1]_q \cdots [1]_q$, $[0]_q^! = 1$ and $G_{[n]}(x) = \frac{G_n(x)}{[n]_q^!}$.

Definition 2.4. A q-difference module over $K(x)$ is said to be of *type G (or a G-q-difference module)* if the following *global q-Galočkin condition* is verified:

$$\sigma_{\mathcal{C}}^q(\mathcal{M}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{v \in \mathcal{C}} \log^+ \left(\sup_{s \leq n} \left|G_{[s]}\right|_{v,Gauss} \right) < \infty\,,$$

where

$$\left| \frac{\sum a_i x^i}{\sum b_j x^j} \right|_{v,Gauss} = \frac{\sup |a_i|_v}{\sup |b_j|_v}\,,$$

for all $\frac{\sum a_i x^i}{\sum b_j x^j} \in K(x)$.

Remark 2.5. Notice that the definition of G-q-difference module involves only the cyclotomic places.

Proposition 2.6. *The definition of G_q-module is independent on the choice of the basis and is stable by extension of scalars to $K'(x)$, for a finite extension K' of K.*

Proof. Once again the proof if similar to the classical theory of G-functions and differential modules of type G. □

3. Role of the "noncyclotomic" places

Proposition 3.1. *In the notation introduced above, for any q-difference module $\mathcal{M} = (M, \Sigma_q)$ over $K(x)$ we have:*

$$\sigma_{\mathcal{P}_f \smallsetminus \mathcal{C}}^{(q)}(\mathcal{M}) := \limsup_{n\to\infty} \frac{1}{n} \sum_{v \in \mathcal{P}_f \smallsetminus \mathcal{C}} \log^+ \left(\sup_{s \leq n} \left|G_{[s]}\right|_v \right) < \infty\,.$$

Proof. We recall that the sequence of matrices $G_{[n]}$ satisfies the recurrence relation:

$$G_{[n+1]}(x) = \frac{G_{[n]}(qx)G_1(x) + d_q G_{[n]}(x)}{[n+1]_q}\,.$$

Since $|[n+1]_q|_v = 1$ for any $v \in \mathcal{P}_f \smallsetminus \mathcal{C}$, we conclude recursively that

$$\left|G_{[n]}\right|_{v,Gauss} \leq 1\,,$$

for almost all places $v \in \mathcal{P}_f \smallsetminus \mathcal{C}$. For the remaining finitely many places $v \in \mathcal{P}_f$, one can deduce from the recursive relation there exists a constant $C > 0$ such that $\left|G_{[n]}\right|_{v,Gauss} \leq C^n$. □

We immediately obtain the equivalence of our definition of q-difference module of type G with the naive analogue of the classical definition of G-module (*cf.* [And89, IV, 4.1]):

Corollary 3.2. *A q-difference module is of type G if and only if*

$$\sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M}) := \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\mathcal{P}_f} \log^+ \left(\sup_{s\leq n} \left|G_{[s]}\right|_v \right) < \infty \,.$$

We expect the same kind of result to be true for G_q-functions, namely:

Conjecture 3.3. *Suppose that* $y = \sum_{n\geq 0} y_n x^n \in K[[x]]$ *is solution of a q-difference equations with coefficients in* K *(*cf. (2.1.1)*). Then:*

$$\sigma_{\mathcal{P}_f \smallsetminus \mathcal{C}}(y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\mathcal{P}_f\smallsetminus\mathcal{C}} \log^+ \left(\sup_{s\leq n} |y_s|_v \right) < \infty \,.$$

The last statement would immediately imply that one can define G_q-functions in the following way:

Conjectural definition 3.4. *We say that the series* $y = \sum_{n\geq 0} y_n x^n \in K[[x]]$ *is a* G_q*-function if* y *is solution of a q-difference equations with coefficients in* K *and moreover*

$$\sigma_{\mathcal{C}\cup\mathcal{P}_\infty}(y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{v\in\mathcal{C}\cup\mathcal{P}_\infty} \log^+ \left(\sup_{s\leq n} |y_s|_v \right) < \infty \,.$$

Remark 3.5. The fact that for almost all $v \in \mathcal{P}_f \smallsetminus \mathcal{C}$ we have $|G_{[n]}(x)|_{v,Gauss} \leq 1$ for any $n \geq 1$ implies that for almost all $v \in \mathcal{P}_f \smallsetminus \mathcal{C}$ a "solution" $y(x) = \sup_n y_n x^n \in K[[x]]$ of a q-difference system with coefficient in $K(x)$ is bounded, in the sense that $\sup_n |y_n|_v < \infty$. Unfortunately, one would need some uniformity with respect to v and n to conclude something about $\sigma_{\mathcal{P}_f\smallsetminus\mathcal{C}}(y)$.

Notice that if 0 is an ordinary point, the conjecture is trivial since

$$\sum_{n\geq 0} G_{[n]}(0) x^n$$

is a fundamental solution of the linear system $Y(qx) = A_1(x)Y(x)$. A q-analogue of the techniques developed in [And89, V] (*cf.* also [DGS94, Chap. VII]) would probably allow to establish the conjecture under the assumption that 0 is a regular point. This is not satisfactory because one of the purposes of the whole theory is the possibility of reading the regularity of a q-difference equation on one single solution (*cf.* Theorem 4.1 below), so one does not want to assume regularity *a priori*.

4. Main results

A q-difference module (M, Σ_q) is said to be regular singular at 0 if there exists a basis $\underline{e}$ such that the Taylor expansion of the matrix $A_1(x)$ is in $Gl_\nu(K[[x]])$. It is said to be regular singular *tout court* if it is regular singular both at 0 and at ∞. We have the following analogue of a well-known differential result (*cf.* [Kat70, §13]; *cf.* also [DV02, §6.2.2] for q-difference modules over a number field):

Theorem 4.1. *A G-q-difference module* $\mathcal{M}$ *over* $K(x)$ *is regular singular.*

Let $\vec{y}(x) = {}^t(y_0(x), \ldots, y_{\nu-1}(x)) \in K[[x]]^\nu$ be a solution of the q-difference system associated to $\mathcal{M} = (M, \Sigma_q)$ with respect to the basis $\underline{e}$:

$$\vec{y}(qx) = A_1(x)\vec{y}(x) \,.$$

We say that $\vec{y}(x)$ is an injective solution if $y_1(x), \ldots, y_\nu(x)$ are lineairly independent over $K(x)$.

We have the following q-analogue of the André-Chudnovsky Theorem [And89, VI]:

Theorem 4.2. *Let* $\vec{y}(x) = {}^t(y_0(x), \ldots, y_{\nu-1}(x)) \in K[[x]]^\nu$ *be an injective solution of the q-difference system associated to* $\mathcal{M} = (M, \Sigma_q)$ *with respect to the basis* $\underline{e}$*.*

If $y_0(x), \ldots, y_{\nu-1}(x)$ *are* G_q*-functions, then* $\mathcal{M}$ *is a G-q-difference module.*

We can immediately state a corollary:

Corollary 4.3. *Let* $\vec{y}(x) = {}^t(y_0(x), \ldots, y_{\nu-1}(x)) \in K[[x]]^\nu$ *be an injective solution of the q-difference system associated to* $\mathcal{M} = (M, \Sigma_q)$ *with respect to the basis* $\underline{e}$*.*

If $y_1(x), \ldots, y_\nu(x)$ *are* G_q*-functions, then* $\mathcal{M}$ *is regular singular.*

Thanks to the cyclic vector lemma we can state the following (*cf.* [Sau00, Annexe B]):

Corollary 4.4. *Let $y(x)$ a G_q-function and let*

$$a_0(x)y(x)+a_1(x)y(qx)+\cdots+a_\nu(x)y(q^\nu x)=0\,. \tag{4.4.1}$$

a q-difference equation of minimal order ν, having $y(x)$ as a solution.

Then (4.4.1) *is fuchsian,* i.e. *we have* $\mathrm{ord}_x a_i \geq \mathrm{ord}_x a_0 = \mathrm{ord}_x a_\nu$ *and* $\deg_x a_i \leq \deg_x a_0 = \deg_x a_\nu$*, for any* $i=0,\dots,\nu$.

The proofs of Theorem 4.1 and Theorem 4.2 are the object of §6 and §7, respectively.

5. Nilpotent reduction at cyclotomic places

We denote by $\mathcal{O}_K$ the ring of integers of K, k_v the residue field of K with respect to the pace v, ϖ_v the uniformizer of v and q_v the image of q in k_v, which is defined for all places $v\in\mathcal{P}$. Notice that q_v is a root of unity for all $v\in\mathcal{C}$. Let $\kappa_v\in\mathbb{N}$ be the order of q_v, for $v\in\mathcal{C}$.

Let $\mathcal{M}=(M,\Sigma_q)$ be a q-difference module over $K(x)$. We can always choose a lattice $\widetilde{M}$ of M over an algebra of the form

$$\mathcal{A}=\mathcal{O}_K\left[x,\frac{1}{P(x)},\frac{1}{P(qx)},\frac{1}{P(q^2x)},\dots\right], \tag{5.0.2}$$

for some $P(x)\in\mathcal{O}_K[x]$, such that for almost all $v\in\mathcal{C}$ we can consider the q_v-difference module $M_v=\widetilde{M}\otimes_{\mathcal{A}}k_v(x)$, with the structure induced by Σ_q. In this way, for almost all $v\in\mathcal{C}$, we obtain a q_v-difference module $\mathcal{M}_v=(M_v,\Sigma_{q_v})$ over $k_v(x)$, having the particularity that q_v is a root of unity. This means that $\sigma_{q_v}^{\kappa_v}=1$ and that $\Sigma_{q_v}^{\kappa_v}$ is a $k_v(x)$-linear operator.

The results in [DV02, §2] apply to this situation: we recall some of them. Since we have:

$$\sigma_{q_v}^{\kappa_v}=1+(q-1)^{\kappa_v}x^{\kappa_v}d_{q_v}^{\kappa_v}$$

and

$$\Sigma_{q_v}^{\kappa_v}=1+(q-1)^{\kappa_v}x^{\kappa_v}\Delta_{q_v}^{\kappa_v}\,,$$

where $\Delta_{q_v}=\frac{\Sigma_{q_v}-1}{(q_v-1)x}$, the following facts are equivalent:

(1) $\Sigma_{q_v}^{\kappa_v}$ is unipotent;
(2) $\Delta_{q_v}^{\kappa_v}$ is a linear nilpotent operator;
(3) the reduction of $A_{\kappa_v}(x)-1$ modulo ϖ_v is a nilpotent matrix;
(4) the reduction of $G_{\kappa_v}(x)$ modulo ϖ_v is nilpotent;
(5) there exists $n\in\mathbb{N}$ such that $|G_{n\kappa_v}(x)|_{v,Gauss}\leq|\varpi_v|_v$.

Definition 5.1. If the conditions above are satisfied we say that $\mathcal{M}$ has *nilpotent reduction (of order n)* modulo $v\in\mathcal{C}$.

Remark 5.2. If the characteristic of k is 0 and if $|G_{\kappa_v}(x)|_{v,Gauss}\leq|[\kappa_v]_q|_v$, the module $\mathcal{M}_v$ has a structure of iterated q-difference module, in the sense of [Har07, §3]. In particular, if v is a non ramified place of $K/k(q)$, then $|[\kappa_v]_q|_v=|\varpi_v|_v$.

The following result is a q-analogue of a well-known differential p-adic estimate (*cf.* for instance [DGS94, page 96]). It has already been proved in the case of q-difference equations over a p-adic field in [DV02, §5.1]. We are only sketching the argument: only the estimate of the q-factorials are slightly different from the case of mixed characteristic.

Proposition 5.3. *If $\mathcal{M}=(M,\Sigma_q)$ has nilpotent reduction(of order n) modulo $v\in\mathcal{C}$ then*

$$\limsup_{m\to\infty}\sup\left(1,\left|G_{[m]}\right|_{v,Gauss}\right)^{1/m}\leq|\varpi_v|_v^{1/n\kappa_n}|[\kappa_v]_q|_v^{-1/\kappa_v}\,.$$

Proof. The Leibniz formula (*cf.* [DV02, Lemma 5.1.2] for a detailed proof in a quite similar situation) implies that for any $s\in\mathbb{N}$ we have:

$$|G_{sn\kappa_v}(x)|_{v,Gauss}\leq|\varpi_v|_v^s\,.$$

Since $|G_1(x)|_{v,Gauss}\leq 1$, for any $m\in\mathbb{N}$ we have:

$$\left|G_{[m]}(x)\right|_{v,Gauss}\leq\frac{\left|G_{[\frac{m}{n\kappa_v}]n\kappa_v}(x)\right|_{v,Gauss}}{|[m]_q^!|_v}\leq\frac{|\varpi_v|_v^{[\frac{m}{n\kappa_v}]}}{|[m]_q^!|_v}\,,$$

where $\left[\frac{m}{n\kappa_v}\right]=\max\{a\in\mathbb{Z}:a\leq\frac{m}{n\kappa_v}\}$. The following lemma on the estimate of $[m]_q^!$ allows to conclude. □

Lemma 5.4. *For $v\in\mathcal{C}$ we have $|[m]_q|_v=|[\kappa]_q|_v$ if $\kappa_v|m$ and $|[m]_q|_v=1$ otherwise. Therefore:*

$$\lim_{m\to\infty}|[m]_q^!|_v^{1/m}=|[\kappa_v]_q|_v^{1/\kappa_v}\,.$$

Proof. Let $m\geq 2$ and $m=s\kappa_v+r$, with $r,s\in\mathbb{Z}$ and $0\leq r<\kappa_v$. If κ_v does not divide m, *i.e.* if $r>0$, we have

$$[m]_q=1+q+\cdots+q^{m-1}=[\kappa_v]_q+q^{\kappa_v}[\kappa_v]_q+\cdots+q^{s\kappa_v}(1+q+\cdots+q^{r-1})\,.$$

Therefore $|[m]_q|_v=1$. On the other hand, if $r=0$:

$$[m]_q=\left(1+q^{\kappa_v}+\cdots+q^{\kappa_v(s-1)}\right)[\kappa_v]_q\,.$$

Since $q^{\kappa_v}\equiv 1$ modulo ϖ_v, we deduce that $1+q^{\kappa_v}+\cdots+q^{\kappa_v(s-1)}\equiv s$ modulo ϖ_v. Therefore

$$\left|[m]_q\right|_v=|s|_v\left|[\kappa_v]_q\right|_v=\left|[\kappa_v]_q\right|_v\,.$$

This implies that

$$\left|[m]_q^!\right|_v=|\kappa_v]_q|_v^{[\frac{m}{\kappa_v}]}\,,$$

which allows to calculate the limit. □

We obtain the following characterization:

Corollary 5.5. *The q-difference module $\mathcal{M}=(M,\Sigma_q)$ has nilpotent reduction modulo $v\in\mathcal{C}$ if and only if*

(5.5.1)
$$\limsup_{m\to\infty}\sup\left(1,\left|G_{[m]}\right|_{v,Gauss}\right)^{1/m}<|[\kappa_v]_q|_v^{-1/\kappa_v}\,.$$

Proof. One side of the implication is an immediate consequence of the proposition above. On the other hand, the assumption (5.5.1) implies that

$$\limsup_{m\to\infty}\sup\left(1,|G_m|_{v,Gauss}\right)^{1/m}<1\,,$$

which clearly implies that there exists n such that $|G_{n\kappa_v}|_{v,Gauss}\leq|\varpi_v|_v$. □

We finally obtain the following proposition, that will be useful in the proof of Theorem 4.1:

Proposition 5.6. *Let $\mathcal{M}$ be q-difference module over $K(x)$ of type G. Let $\mathcal{C}_0$ be the set of $v\in\mathcal{C}$ such that $\mathcal{M}$ does not have nilpotent reduction modulo v. Then*

$$\sum_{v\in\mathcal{C}_0}\frac{1}{\kappa_v}<+\infty\,.$$

In particular, $\mathcal{M}$ has nilpotent reduction modulo v for infinitely many $v\in\mathcal{C}$.

The proof relies on the following lemma:

Lemma 5.7. *The following limit exists:*

$$\lim_{n\to\infty}\frac{1}{n}\log^+\left(\sup_{s\leq n}\left|G_{[s]}(x)\right|_{v,Gauss}\right)\,.$$

Proof. The proof is essentially the same as the proof of [DV02, 4.2.7], a part from the estimate of the q-factorials (*cf.* Lemma 5.4 above). The key point is the following formula:

$$G_{[n+s]}(x)=\sum_{i+j=n}\frac{[n]_q^![s]_q^!}{[s+n]^!}\frac{d_q^j}{[j]_q^!}\left(G_{[s]}(q^ix)\right)G_{[i]}(x)\,,\;\forall\, s,n\in\mathbb{N}\,,$$

obtained iterating the Leibniz rule. □

Proof of Proposition 5.6. The Fatou lemma, together with Lemma 5.7, implies:

$$\sum_{v\in\mathcal{C}}\lim_{n\to\infty}\frac{1}{n}\log^+\left(\sup_{s\le n}\left|G_{[s]}(x)\right|_{v,Gauss}\right)\le\liminf_{n\to\infty}\frac{1}{n}\sum_{v\in\mathcal{C}}\log^+\left(\sup_{s\le n}\left|G_{[s]}(x)\right|_{v,Gauss}\right)\le\sigma_{\mathcal{C}}^{(q)}(\mathcal{M})<\infty\,.$$

It follows from Corollary 5.5 that:

$$\sum_{v\in\mathcal{C}_0}\frac{\log^+|[\kappa_v]_q|_v^{-1}}{\kappa_v}<\infty$$

and hence that

$$\sum_{v\in\mathcal{C}_0}\frac{\log d^{-1}}{\kappa_v}<\infty\,,$$

since only a finite number of places of $K/k(q)$ are ramified. □

6. Proof of Theorem 4.1

It is enough to prove that 0 is a regular singular point for $\mathcal{M}$, the proof at ∞ being completely analogous.

Let $r\in\mathbb{N}$ be a divisor of $\nu!$ and let L be a finite extension of K containing an element $\widetilde{q}$ such that $\widetilde{q}^r=q$. We consider the field extension $K(x)\hookrightarrow L(t)$, $x\mapsto t^r$. The field $L(t)$ has a natural structure of $\widetilde{q}$-difference algebra extending the q-difference structure of $K(x)$. Remark that:

Lemma 6.1. *The q-difference module $\mathcal{M}$ is regular singular at $x=0$ if and only if the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}:=(M\otimes_{K(t)}L(t),\Sigma_{\widetilde{q}}:=\Sigma_q\otimes\sigma_{\widetilde{q}})$ is regular singular at $t=0$.*

Proof. It is enough to notice that if $\underline{e}$ is a cyclic basis for $\mathcal{M}$, then $\underline{e}\otimes 1$ is a cyclic basis for $\mathcal{M}_{L(t)}$ and $\Sigma_{\widetilde{q}}(\underline{e}\otimes 1)=\Sigma_q(\underline{e})\otimes 1$. □

The next lemma can be deduced from the formal classification of q-difference modules (*cf.* [Pra83, Cor. 9 and §9, 3)], [Sau04, Thm. 3.1.7]):

Lemma 6.2. *There exist an extension $L(t)/K(x)$ as above, a basis $\underline{f}$ of the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$, such that $\Sigma_{\widetilde{q}}\underline{f}=\underline{f}B(t)$, with $B(t)\in Gl_\mu(L(t))$, and an integer ℓ such that*

$$\left\{\begin{array}{l}B(t)=\dfrac{B_\ell}{t^\ell}+\dfrac{B_{\ell-1}}{t^{\ell-1}}+\dots\,,\text{ as an element of }Gl_\mu(L((t)));\\ B_\ell\text{ is a constant non nilpotent matrix.}\end{array}\right.\tag{6.2.1}$$

Proof of Theorem 4.1. Let $\mathcal{B}\subset L(t)$ be a $\widetilde{q}$-difference algebra over the ring of integers $\mathcal{O}_L$ of L, of the same form as (5.0.2), containing the entries of $B(t)$. Then there exists a $\mathcal{B}$-lattice $\mathcal{N}$ of $\mathcal{M}_{L(t)}$ inheriting the $\widetilde{q}$-difference module structure from $\mathcal{M}_{L(t)}$ and having the following properties:

1. $\mathcal{N}$ has nilpotent reduction modulo infinitely many cyclotomic places of L;
2. there exists a basis $\underline{f}$ of $\mathcal{N}$ over $\mathcal{B}$ such that $\Sigma_{\widetilde{q}}\underline{f}=\underline{f}B(t)$ and $B(t)$ verifies (6.2.1).

Iterating the operator $\Sigma_{\widetilde{q}}$ we obtain:

$$\Sigma_{\widetilde{q}}^m(\underline{f})=\underline{f}B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{m-1}t)=\underline{f}\left(\frac{B_\ell^m}{\widetilde{q}^{\frac{\ell m(\ell m-1)}{2}}x^{m\ell}}+h.o.t.\right)$$

We know that for infinitely many cyclotomic places w of L, the matrix $B(t)$ verifies

$$\left(B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{\kappa_w-1}t)-1\right)^{n(w)}\equiv 0\bmod\varpi_w\,,\tag{6.2.2}$$

where ϖ_w is an uniformizer of the place w, κ_w is the order $\widetilde{q}$ modulo ϖ_w and $n(w)$ is a convenient positive integer. Suppose that $\ell \neq 0$. Then $B_\ell^{\kappa_w} \equiv 0$ modulo ϖ_w, for infinitely many w, and hence B_ℓ is a nilpotent matrix, in contradiction with lemma 6.2. So necessarily $\ell = 0$.

Finally we have $\Sigma_{\widetilde{q}}(\underline{f}) = \underline{f}\,(B_0 + h.o.t)$. It follows from (6.2.1) that B_0 is actually invertible, which implies that $\mathcal{M}_{L(t)}$ is regular singular at 0. Lemma 6.1 allows to conclude. □

7. Proof of Theorem 4.2

7.1. **Idea of the proof.** The hypothesis states that there exists a vector $\vec{y} = {}^t(y_0, \dots, y_{\nu-1}) \in K[[x]]^\nu$, which is solution of the q-difference system:

$$\vec{y}(qx) = A_1(x)\vec{y}(x)\,, \tag{7.0.3}$$

and therefore of the systems $d_q^n \vec{y} = G_n(x)\vec{y}$ and $\sigma_q^n \vec{y} = A_n(x)\vec{y}$ for any $n \geq 1$, having the property that $y_0, \dots, y_{\nu-1}$ are linearly independent over $K(x)$. We recall that

$$G_{n+1}(x) = G_n(qx)G_1(x) + d_q G_n(x)$$

and that

$$A_{n+1}(x) = A_n(qx)A_1(x)\,.$$

Let us consider the operator:

$$\Lambda = A_1(x)^{-1} \circ (d_q - G_1(x))\;.$$

We know that there exists an extension $\mathcal{U}$ of $K(x)$ (for instance the universal Picard-Vessiot ring constructed in [vdPS97, §12.1]) such that we can find an invertible matrix $\mathcal{Y}$ with coefficient in $\mathcal{U}$ solution of our system $d_q\mathcal{Y} = G_1\mathcal{Y}$. An explicit calculation shows that:

$$d_q \circ \mathcal{Y}^{-1} = (\sigma_q\mathcal{Y})^{-1}\,(d_q - G_1(x)) = \mathcal{Y}^{-1}A_1(x)^{-1}\,(d_q - G_1(x))$$

and therefore that:

$$\Lambda^n = \mathcal{Y} \circ d_q^n \circ \mathcal{Y}^{-1}\,,\ \text{for all integers } n \geq 0. \tag{7.0.4}$$

We set $\binom{n}{i}_q = \frac{[n]_q^!}{[i]_q^![n-1]_q^!}$, for any pair of integers $n \geq i \geq 0$. The twisted q-binomial formula shows that $\left|\binom{n}{i}_q\right|_v \leq 1$ for any $v \in \mathcal{P}_f$.

The proof of Theorem 4.2 is based on the following q-analogue of [And89, VI, §1]:

Proposition 7.1. *There exist $\alpha_0^{(n)}, \dots, \alpha_n^{(n)} \in K$ such that for all $\vec{P} \in K[x]^\nu$ and all $n \geq 0$ we have:*

$$G_{[n]}\vec{P} = \sum_{i=0}^{n} \frac{(-1)^i}{[n]_q^!}\binom{n}{i}_q \alpha_i^{(n)} d_q^{n-i} \circ A_i(x)\Lambda^i(\vec{P})\,, \tag{7.1.1}$$

with $|\alpha_i(n)|_v \leq 1$, for any $v \in \mathcal{P}_f$ and $n \geq i \geq 0$.

Proof. The iterated twisted Leibniz Formula (*cf.* for instance [DV02, 1.1.8.1])

$$d_q^n(fg) = \sum_{j=0}^{n}\binom{n}{j}_q \sigma_q^j\left(d_q^{n-j}(f)\right) d_q^j(g)\,,\ \forall f,g \in \mathcal{U}$$

implies

$$\begin{aligned}
&\sum_{i=0}^{n}\frac{(-1)^i}{[n]_q^!}\binom{n}{i}_q\alpha_i^{(n)}d_q^{n-i}\circ A_i(x)\circ\Lambda^i(\vec{P})\\
&\quad=\sum_{i=0}^{n}\frac{(-1)^i}{[n]_q^!}\binom{n}{i}_q\alpha_i^{(n)}d_q^{n-i}\circ\sigma_q^i(\mathcal{Y})\circ d_q^i\circ\mathcal{Y}^{-1}(\vec{P})\\
&\quad=\sum_{i=0}^{n}\frac{(-1)^i}{[n]_q^!}\binom{n}{i}_q\alpha_i^{(n)}\sum_{j=0}^{n-i}\binom{n-i}{j}_q q^{ij}\sigma_q^{n-j}\left(d_q^j(\mathcal{Y})\right)\circ d_q^{n-j}\circ\mathcal{Y}^{-1}(\vec{P})\\
&\quad=\sum_{j=0}^{n}\left(\sum_{i=0}^{n-j}\frac{(-1)^i}{[n]_q^!}\binom{n}{i}_q\binom{n-i}{j}_q q^{ij}\alpha_i^{(n)}\right)\sigma_q^{n-j}\left(d_q^j(\mathcal{Y})\right)\circ d_q^{n-j}\circ\mathcal{Y}^{-1}(\vec{P})\\
&\quad=\sum_{j=0}^{n}\frac{1}{[n-j]_q^![j]_q^!}\left(\sum_{i=0}^{n-j}(-1)^i\binom{n-j}{i}_q q^{ij}\alpha_i^{(n)}\right)\sigma_q^{n-j}\left(d_q^j(\mathcal{Y})\right)\circ d_q^{n-j}\circ\mathcal{Y}^{-1}(\vec{P})\,.
\end{aligned}$$

We have to solve the linear system:

$$\sum_{i=0}^{n-j}(-1)^i\binom{n-j}{i}_q q^{ij}\alpha_i^{(n)}=\begin{cases}1 & \text{if } n=j,\\ 0 & \text{otherwise.}\end{cases}$$

For $n=j$ we obtain $\alpha_0^{(n)}=1$. We suppose that we have already determined $\alpha_0^{(n)},\dots,\alpha_{k-1}^{(n)}$. For $n-j=k$ we get:

$$\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}_q q^{i(n-k)}\alpha_i^{(n)}=(-1)^{k+1}\alpha_k^{(n)}q^{k(n-k)}\,.$$

This proves also that $|\alpha_k^{(n)}|_v\leq 1$ for ant $v\in\mathcal{P}_f$. □

For all $\vec{P}={}^t(P_0,\dots,P_\nu-1)\in K[x]^\nu$ and $n\geq 0$ we set:

$$\vec{R}_n=\frac{\Lambda^n}{[n]_q^!}(\vec{P})$$

and:

$$R^{<n>}=\left(\binom{n}{n}_q\vec{R}_n\quad\binom{n+1}{n}_q\vec{R}_{n+1}\dots\quad\binom{n+\nu-1}{n}_q\vec{R}_{n+\nu-1}\right)\,.$$

Therefore we obtain the identity:

Corollary 7.2.

$$G_{[n]}R^{<0>}=\sum_{i=0}^{n}(-1)^i\alpha_i^{(n)}\frac{d_q^{n-i}}{[n-i]_q^!}\circ A_i(x)R^{<i>}$$

Remark 7.3. In order to obtain an estimate of $\sigma_{\mathcal{P}_f}^{(q)}(\mathcal{M})$ we want to estimate the matrices $G_{[n]}(x)$. The main point of the proof is the construction of a vector $\vec{P}$, linked to the solution vector $\vec{y}$ of (7.0.3), such that $R^{<0>}$ is an invertible matrix.

The proof is divided in step: in step 1 we construct $\vec{P}$; in step 2 we prove that $R^{<0>}$ is invertible; step 3 and 4 are devoted to the estimate of $G_{[n]}(x)$ and of $\sigma_{\mathcal{P}_f}^{(q)}(\mathcal{M})$.

7.2. **Step 1. Hermite-Padé approximations of $\vec{y}$.** We denote by deg the usual degree in x and by ord the order at $x=0$. We extend their definitions to vectors as follows:

$$\deg\vec{P}(x)=\sup_{i=0,\dots,\nu-1}\deg P_i(x)\,,\ \text{for all } \vec{P}={}^t\left(P_0(x),\dots,P_{\nu-1}(x)\right)\in K[x]^\nu.$$

$$\mathrm{ord}\vec{P}(x)=\inf_{i=0,\dots,\nu-1}\mathrm{ord}P_i(x)\,,\ \text{for all } \vec{P}={}^t\left(P_0(x),\dots,P_{\nu-1}(x)\right)\in K((x))^\nu.$$

Moreover we set:

$$\begin{cases} \left(\sum_{n\geq 0}\vec{a}_n x^n\right)_{\leq N} = \sum_{n\leq N}\vec{a}_n x^n\,, \\ \\ \left(\sum_{n\geq 0}\vec{a}_n x^n\right)_{>N} = \sum_{n>N}\vec{a}_n x^n\,, \end{cases} \qquad \text{for all } \sum_{n\geq 0}\vec{a}_n x^n \in K[[x]]^\nu\,.$$

Finally, for $g(x)=\sum_{n\geq 0} g_n x^n \in K[x]$ and for $\vec{y}=\sum_{n\geq 0}\vec{y}_n x^n \in K[[x]]^\nu$ we set:

$$h(g,v)=\sup_n \log^+ |g_n|_v\,,\ \forall\ v\in\mathcal{P}\,,$$

$$h(g)=\sum_{v\in\mathcal{P}} h(g,v)$$

and

$$\widetilde{h}(n,v)=\sup_{s\leq n}\log^+|\vec{y}_s|_v\,,\ \forall\ v\in\mathcal{P}\,,$$

where $|\vec{y}_s|_v$ is the maximum of the v-adic absolute value of the entries of $\vec{y}_s$.

The following lemma is proved in [And89, VI, §3] or [DGS94, Chap. VIII,§3] in the case of a number field. The proof in the present case is exactly the same, apart from the fact that there are no archimedean places in $\mathcal{P}$:

Proposition 7.4. *Let $\tau\in(0,1)$ be a constant and $\vec{y}=\sum_{n\geq 0}\vec{y}_n x^n\in K[[x]]^\nu$. For all integers $N>0$ there exists $\vec{g}(x)\in K[x]^\nu$ having the following properties:*

(7.4.1)
$$\deg g(x)\leq N\,;$$

(7.4.2)
$$\mathrm{ord}(g\vec{y})_{\leq N}\geq 1+N+\left[N\frac{1-\tau}{\nu}\right]\,;$$

(7.4.3)
$$h(g)\leq const+\frac{1-\tau}{\tau}\sum_{v\in\mathcal{P}}\widetilde{h}\left(N+\left[N\frac{1-\tau}{\nu}\right],v\right)\,.$$

From now on we will assume that $\vec{P}(x)=(g\vec{y})_{\leq N}$.

Proposition 7.5. *Let $Q_1(x)\in\mathcal{V}_K[x]$ be a polynomial such that $Q_1(x)A_1^{-1}(x)\in M_{\nu\times\nu}(K[x])$. We set:*

$$Q_0=1 \text{ and } Q_n(x)=Q_1(x)Q_{n-1}(qx)\,, \text{ for all } n\geq 1,$$

and

$$t=\sup\left(\deg(Q_1(x)A_1^{-1}(x)),\deg Q_1(x)\right)\,.$$

If $n\leq\frac{N}{t}\frac{1-\tau}{\nu}$, then

$$\left(x^nQ_n(x)\frac{d_q^n g}{[n]_q^!}(x)\vec{y}(x)\right)_{\leq N+nt}=x^nQ_n(x)\vec{R}_n\,.$$

The proposition above is a consequence of the following lemmas:

Lemma 7.6. *For each $n\geq 0$ we have:*

(7.6.1)
$$x^nQ_n(x)\vec{R}_n(x)\in K[x]^\nu\,;$$

(7.6.2)
$$\deg x^nQ_n(x)\vec{R}_n(x)\leq N+nt\,.$$

Proof. Clearly $\vec{R}_0=(g\vec{y})_{\leq N}\in K[x]^\nu$. We recall that there exist $c_{i,n}\in K$ such that (*cf.* [DV02, 1.1.10]):

$$d_q^n=\frac{(-1)^n}{(q-1)^nx^n}(\sigma_q-1)(\sigma_q-q)\cdots(\sigma_q-q^{n-1})=\frac{(-1)^n}{(q-1)^nx^n}\sum_{i=1}^n c_{i,n}\sigma_q^i\,,$$

for each $n \geq 1$. Therefore we obtain:

$$\begin{array}{rcl} x^n Q_n(x)\vec{R}_n & = & x^n Q_n(x)\mathcal{Y}\frac{d_q^n}{[n]_q^!}\left(\mathcal{Y}^{-1}\vec{P}\right) \\ & = & \frac{Q_n(x)\mathcal{Y}}{[n]_q^!(q-1)^n}\sum_{i=0}^{n} c_{i,n}\sigma_q^i\left(\mathcal{Y}^{-1}\vec{P}\right) \\ & = & \frac{1}{[n]_q^!(q-1)^n}\sum_{i=0}^{n} c_{i,n}Q_n(x)A_i^{-1}(x)\sigma_q^i(\vec{P})\,. \end{array}$$

Since $A_i(x) = A_1(q^{i-1}x)\cdots A_1(x)$, we conclude that $x^n Q_n(x)\vec{R}_n \in K[x]^\nu$ and:

$$\begin{array}{rcl} \deg x^n Q_n(x)\vec{R}_n & \leq & \sup_{i=0,\dots,n} \deg\left(Q_n(x)A_i^{-1}(x)\sigma_q^i(\vec{P})\right) \\ & \leq & \sup_{i=0,\dots,n}\left(\deg(Q_i(x)A_i^{-1}(x)) + \deg Q_{n-i}(q^i x) + \deg\sigma_q^i(\vec{P})\right) \\ & \leq & N + nt\,. \end{array}$$

□

Lemma 7.7.

$$\mathrm{ord}\left(x^n Q_n(x)\frac{d_q^n(g)}{[n]_q^!}(x)\vec{y}(x) - x^n Q_n(x)\vec{R}_n\right) \geq 1 + N + \left[N\frac{1-\tau}{\nu}\right]\,.$$

Proof. We have:

$$\begin{array}{l} x^n Q_n(x)\frac{d_q^n(g)}{[n]_q^!}(x)\vec{y}(x) - x^n Q_n(x)\vec{R}_n \\ = \frac{1}{[n]_q^!(q-1)^n}\sum_{l=0}^{n} c_{l,n}Q_n(x)\left(\sigma_q^l(g(x))\vec{y}(x) - \mathcal{Y}\sigma_q^l\left(\mathcal{Y}^{-1}\vec{P}\right)\right) \\ = \frac{1}{[n]_q^!(q-1)^n}\sum_{l=0}^{n} c_{l,n}Q_n(x)\left(\sigma_q^l(g(x))\vec{y}(x) - A_l^{-1}(x)\sigma_q^l(\vec{P})\right)\,. \end{array}$$

Let $\vec{H}_l = Q_l(x)\sigma_q^l(g(x))\vec{y}(x) - Q_l(x)A_l^{-1}(x)\sigma_q^l(\vec{P})$. Since:

$$A_1^{-1}(x)Q_1(x)\sigma_q\left(\vec{H}_l\right) = \vec{H}_{l+1}\,,$$

by induction on l we obtain:

$$\mathrm{ord}\vec{H}_l \geq \mathrm{ord}\vec{H}_{l-1} \geq \mathrm{ord}\left(g(x)\vec{y}(x) - \vec{P}(x)\right) \geq 1 + N + \left[N\frac{1-\tau}{\nu}\right]\,.$$

□

7.3. **Step 2. The matrix $R^{<0>}$.**

Theorem 7.8. *Let $\vec{y}(x) = {}^t(y_0(x),\dots,y_{\nu-1}(x)) \in K[[x]]^\nu$ a solution vector of $\Lambda Y = 0$, such that $y_0(x),\dots,y_{\nu-1}(x)$ are linearly independent over $K(x)$. Then there exists a constant $C(\Lambda)$, depending only on Λ, such that if*

$$\vec{P} = {}^t(P_0\dots,P_{\nu-1}) \in K[x]^\nu \smallsetminus \{\underline{0}\}$$

has the following property:

(7.8.1) $$\mathrm{ord}\det\begin{pmatrix} P_i & P_j \\ y_i & y_j \end{pmatrix} \geq \deg\vec{P}(x) + C(\Lambda)\,, \forall i,j = 0,\dots,\nu-1,$$

then the matrix $R^{<0>}$ is invertible.

Remark 7.9. We remark that if we choose g as in Propositions 7.4 and 7.5 and $\vec{P} = (g\vec{y})_{\leq N}$, for $N >> 0$ we have:

$$N\frac{1-\tau}{\nu} \geq C(\Lambda)\,.$$

Therefore the condition (7.8.1) is satisfied since:

$$\operatorname{ord}\det\begin{pmatrix}P_i & P_j\\ y_i & y_j\end{pmatrix}=\operatorname{ord}\det\begin{pmatrix}(gy_i)_{>N} & (gy_j)_{>N}\\ y_i & y_j\end{pmatrix}\geq 1+N+N\frac{1-\tau}{\nu}\,.$$

We recall the Shidlovsky's Lemma that we will need on the proof of Theorem 7.8.

Definition 7.10. We define total degree of $\frac{f(x)}{g(x)}\in K(x)$ as:

$$\mathrm{deg.tot}\frac{f(x)}{g(x)}=\deg f(x)+\deg g(x)\,.$$

Lemma 7.11 (Shidlovsky's Lemma; *cf.* for instance [DGS94, Chap. VIII, 2.2])**.** *Let $\mathcal{G}/K(x)$ be a field extension and let $V\subset\mathcal{G}$ a K-vector space of finite dimension. Then the total degree of the elements of $K(x)$ that can be written as quotient of two element of V is bounded.*

Proof of the Theorem 7.8. Let $\mathcal{Y}$ be an invertible matrix with coefficients in an extension $\mathcal{U}$ of $K(x)$ such that $\Lambda\mathcal{Y}=0$ and let C be the field of constant of $\mathcal{U}$ with respect to d_q. The matrix

$$R^{<0>}=\mathcal{Y}\left(\mathcal{Y}^{-1}\vec{P},d_q\left(\mathcal{Y}^{-1}\vec{P}\right),\cdots,\frac{d_q^{\nu-1}}{[\nu-1]_q^!}\left(\mathcal{Y}^{-1}\vec{P}\right)\right)$$

is invertible if and only if

$$\operatorname{rank}\left(\mathcal{Y}^{-1}R^{<0>}\right)=\operatorname{rank}\left(\mathcal{Y}^{-1}\vec{P},\sigma_q\left(\mathcal{Y}^{-1}\vec{P}\right),\dots,\sigma_q^{\nu-1}\left(\mathcal{Y}^{-1}\vec{P}\right)\right)$$

is maximal. Let us suppose that

$$\operatorname{rank}\left(\mathcal{Y}^{-1}R^{<0>}\right)=r<\nu\,.$$

Then the q-analogue of the wronskian lemma (*cf.* for instance [DV02, §1.2]) implies that there exists an invertible matrix M with coefficients in C such that the first column of $M\mathcal{Y}^{-1}R^{<0>}$ is equal to:

$$M\mathcal{Y}^{-1}\vec{P}={}^t\left(\widetilde{w}_0,\widetilde{w}_1,\dots,\widetilde{w}_{r-1},0,\dots,0\right)\,.$$

The matrix $\mathcal{Y}M^{-1}$ still verifies the q-difference equation $\Lambda Y=0$, so we will write $\mathcal{Y}$ instead of $\mathcal{Y}M^{-1}$, to simplify notation. We set:

$$\vec{S}_n=\mathcal{Y}\circ\sigma_q^n\circ\mathcal{Y}^{-1}\vec{P}\,,\ \forall n\geq 0\,,$$

$$S^{<0>}=\left(\vec{S}_0,\dots,\vec{S}_{\nu-1}\right)=\begin{pmatrix}S_{IJ} & S_{IJ'}\\ S_{I'J} & S_{I'J'}\end{pmatrix}$$

and

$$\mathcal{Y}^{-1}=\begin{pmatrix}\mathcal{Y}_{JL} & \mathcal{Y}_{JL'}\\ \mathcal{Y}_{J'L} & \mathcal{Y}_{J'L'}\end{pmatrix}\,,$$

where $I=J=L=\{0,1,\dots,r-1\}$ and $I'=J'=L'=\{r,\dots,\nu-1\}$. We have:

$$\begin{pmatrix}\mathcal{Y}_{JL} & \mathcal{Y}_{JL'}\\ \mathcal{Y}_{J'L} & \mathcal{Y}_{J'L'}\end{pmatrix}\begin{pmatrix}S_{IJ} & S_{IJ'}\\ S_{I'J} & S_{I'J'}\end{pmatrix}=\left(\sigma_q^i\left(\mathcal{Y}^{-1}\vec{P}\right)\right)_{i=0,\dots,\nu-1}=\begin{pmatrix}A\\ 0\end{pmatrix}\,,$$

with $A\in M_{r\times\nu}(K(x))$, and therefore:

$$\mathcal{Y}_{J'L}S_{IJ}+\mathcal{Y}_{J'L'}S_{I'J}=0\,.$$

Because of our choice of $\mathcal{Y}$, the vectors $\vec{S}_0,\dots,\vec{S}_{r-1}$ are linearly independent over $K(x)$, so by permutation of the entries of the vector $\vec{P}$ we can suppose that the matrix S_{IJ} is invertible.

Let $B=S_{I'J}S_{IJ}^{-1}$. Since $S^{<0>}\in M_{\nu\times\nu}\left(K(x)\right)$ is independent of the choice of the matrix $\mathcal{Y}$, the same is true for B. The matrix $\mathcal{Y}$ is invertible and

$$\begin{pmatrix}\mathcal{Y}_{J'L} & \mathcal{Y}_{J'L'}\end{pmatrix}=\mathcal{Y}_{J'L'}\begin{pmatrix}-B & I_{\nu-r}\end{pmatrix}\,,$$

therefore the matrix $\mathcal{Y}_{J'L'}$ is also invertible and we have:

$$B=-\mathcal{Y}_{J'L'}^{-1}\mathcal{Y}_{J'L}\,.$$

The coefficients of the matrix B can be written in the form ξ/η, where ξ and η are elements of the K-vector space of polynomials of degree less or equal to $\nu-r$ with coefficients in K in the entries of the matrix $\mathcal{Y}$. By Shidlovsky's lemma the total degree of the entries of the matrix B is bounded by a constant depending only on the q-difference system Λ.

Let us consider the matrices:

$$Q_1 = \begin{pmatrix} y_{\nu-1} & 0 & 0 & \cdots & 0 \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \in M_{r\times r}(K[[x]])$$

and

$$Q_2 = \begin{pmatrix} 0 & \cdots & 0 & -y_0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{r\times \nu-r}(K[[x]])\,;$$

we set:

$$T = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} S_{IJ} \\ S_{I'J} \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} \mathbb{I}_r \\ B \end{pmatrix} S_{IJ}\,.$$

Let $(b_0,\ldots,b_{r-1})$ be the last row of B. We have:

$$\begin{aligned} \det\left(TS_{IJ}^{-1}\right) &= \det\left(Q_1+Q_2B\right) \\ &= \det \begin{pmatrix} y_{\nu-1}-y_0b_0 & -y_0b_1 & -y_0b_2 & \cdots & -y_0b_{r-1} \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \\ &= \det \begin{pmatrix} y_{\nu-1}-y_0b_0-y_1b_1-\cdots-y_{r-1}b_{r-1} & 0 & 0 & \cdots & 0 \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \\ &= (-y_0)^{r-1}\left(y_{\nu-1}-y_0b_0-y_1b_1-\cdots-y_{r-1}b_{r-1}\right)\,. \end{aligned}$$

We notice that $\det\left(TS_{IJ}^{-1}\right)\neq 0$, since by hypothesis $y_0,\ldots,y_{\nu-1}$ are linearly independent over $K(x)$. Our purpose is to find a lower and an upper bound for $\operatorname{ord}\det\left(TS_{IJ}^{-1}\right)$.

Since the total degree of the entries of B is bounded by a constant depending only on Λ, there exists a constant C_1, depending on Λ and not on $\vec{P}$, such that:

$$\operatorname{ord}\det\left(TS_{IJ}^{-1}\right)\leq C_1\,.$$

Now we are going to determine a lower bound. Let:

$$\vec{S}_n = {}^t\left(S_{n,0},S_{n,2},\ldots,S_{n,\nu-1}\right)\,,\ \text{pour tout } n\geq 0;$$

then we have:

$$S^{<0>} = \left(S_{i,j}\right)_{i,j\in\{0,1,\ldots,\nu-1\}}\,;$$

moreover we set:

$$A_1^{-1} = \left(A_{i,j}\right)_{i,j\in\{0,1,\ldots,\nu-1\}}\,.$$

The elements of the first row of T are of the form:

$$\det\begin{pmatrix} y_{\nu-1} & S_{s,\nu-1} \\ y_0 & S_{s,0} \end{pmatrix}\,,\ \text{pour } s=0,\ldots,r-1,$$

and the ones of the i-th row, for $i=1,\ldots,r-1$:

$$\det\begin{pmatrix} y_i & S_{s,i} \\ y_0 & S_{s,0} \end{pmatrix}\,,\ \text{pour } s=0,\ldots,r-1.$$

Since $\vec{S}_{n+1}=A_1(x)^{-1}\sigma_q(\vec{S}_n)$ we have:

$$\det\begin{pmatrix} y_i & S_{s+1,i}\\ y_j & S_{s+1,j}\end{pmatrix}=\det\begin{pmatrix} y_i & \sum_l A_{i,l}\sigma_q(S_{s,l})\\ y_j & \sum_l A_{j,l}\sigma_q(S_{s,l})\end{pmatrix}\,,$$

therefore:

$$\inf_{i,j=0,\dots,\nu-1}\operatorname{ord}\det\begin{pmatrix} y_i & S_{s+1,i}\\ y_j & S_{s+1,j}\end{pmatrix}\geq (s+1)\operatorname{ord}A_1(x)^{-1}+\inf_{i,j=0,\dots,\nu-1}\operatorname{ord}\det\begin{pmatrix} y_i & P_i\\ y_j & P_j\end{pmatrix}\,.$$

Finally,

$$\operatorname{ord}\det T\geq r(\nu-1)\operatorname{ord}A_1(x)^{-1}+r\inf_{i,j=0,\dots,\nu-1}\operatorname{ord}\det\begin{pmatrix} y_i & P_i\\ y_j & P_j\end{pmatrix}\,.$$

By Lemma 7.6, we obtain:

$$\begin{aligned}\operatorname{ord}\det S_{I,J} &\leq \deg(\text{numerator of }\det S_{I,J})\\ &\leq \sum_{i=0}^{r-1}\deg(\text{numerator of }\vec{S}_i)\\ &\leq r\deg\vec{P}+t\frac{r(r-1)}{2}\,.\end{aligned}$$

We deduce that:

$$\begin{aligned}\operatorname{ord}\det\left(TS_{I,J}^{-1}\right) &\geq \operatorname{ord}\det(T)-\operatorname{ord}\det(S_{I,J})\\ &\geq r\left((\nu-1)\operatorname{ord}A_1(x)^{-1}+\inf_{i,j=0,\dots,\nu-1}\operatorname{ord}\det\begin{pmatrix} y_i & P_i\\ y_j & P_j\end{pmatrix}-\deg\vec{P}-t\frac{(r-1)}{2}\right)\\ &\geq r\left(\inf_{i,j}\operatorname{ord}\det\begin{pmatrix} y_i & P_i\\ y_j & P_j\end{pmatrix}-\deg\vec{P}\right)+C_2\,,\end{aligned}$$

where C_2 is a constant depending only on Λ. To conclude it is enough to choose a constant $C(\Lambda)>\frac{C_1-C_2}{r}$. □

7.4. **Step 3. First part of estimates.** We set:

$$\begin{aligned} y&=\sum_{n\geq 0}\vec{y}_n x^n\,,\ \text{with }\vec{y}_n\in K^\nu\,,\\ \sigma_f(\vec{y})&=\limsup_{n\to+\infty}\frac{1}{n}\left(\sum_{v\in\mathcal{P}_f}\sup_{s\leq n}\log^+|\vec{y}_s|_v\right)\,,\\ \sigma_\infty(\vec{y})&=\limsup_{n\to+\infty}\frac{1}{n}\left(\sum_{v\in\mathcal{P}_\infty}\sup_{s\leq n}\log^+|\vec{y}_s|_v\right)\,.\end{aligned}$$

We recall that we are working under the assumption:

$$\sigma(y)=\limsup_{n\to+\infty}\frac{1}{n}\left(\sum_{v\in\mathcal{P}}\widetilde{h}(n,v)\right)=\sigma_f(\vec{y})+\sigma_\infty(\vec{y})<+\infty$$

and that we want to show that $\sigma^{(q)}_{\mathcal{C}}(\mathcal{M})\leq\infty$. Since $\sigma^{(q)}_{\mathcal{C}}(\mathcal{M})\leq\sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M})$, we will rather show that:

$$\sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M})=\limsup_{n\to+\infty}\frac{1}{n}\left(\sum_{v\in\mathcal{P}_f}h(\mathcal{M},n,v)\right)<\infty\,,$$

where:

$$h(\mathcal{M},n,v)=\sup_{s\leq n}\log^+\left|\frac{G_n}{[n]_q^!}\right|_{v,Gauss}\,.$$

In the sequel g will be a polynomial constructed as in Proposition 7.4. For such a choice of g and for $\vec{P}=(g\vec{y})_{\leq N}$, the hypothesis of Corollary 7.2, Proposition 7.4 and Theorem 7.8 are satisfied.

Proposition 7.12. *We have:*

$$\sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M}) \le \sigma_f(\vec{y})\left(\frac{\nu^2 t}{1-\tau}+t\right)+\Omega+\sum_{v\in\mathcal{P}_f}\log^+|A_1(x)|_{v,Gauss}\,,$$

where:

$$\Omega=\limsup_{n\to+\infty}\frac{1}{n}\left(\nu\sum_{v\in\mathcal{P}_f}h(g,v)+\sum_{v\in\mathcal{P}_f}\log^+\left|\left(\prod_{i=1}^{\nu-1}Q_i(x)\right)\Delta(x)\right|^{-1}_{v,Gauss}\right).$$

Proof. We fix $N, n >> 0$ such that:

$$n+\nu-1\le\frac{N}{t}\frac{1-\tau}{\nu}\,. \tag{7.12.1}$$

Proposition 7.5 and Corollary 7.2 implies that for all integers $s \le n+\nu-1$, we have:

$$\left(x^sQ_s(x)\frac{d_q^s g}{[s]_q^!}(x)\vec{y}(x)\right)_{\le N+st}=x^sQ_s(x)\vec{R}_s \tag{7.12.2}$$

and:

$$G_{[}s]=\sum_{i\le s}(-1)^i\alpha_i^{(n)}\frac{d_q^{s-i}}{[s-i]_q^!}\left(A_i(x)R^{<i>}\right)\left(R^{<0>}\right)^{-1}\,.$$

For all $v\in\mathcal{P}_f$ we deduce:

$$\begin{array}{rcl}\left|G_{[}s]\right|_{v,Gauss} & \le & \left(\sup_{i\le s}\left|\frac{d_q^{s-i}}{[s-i]_q^!}(A_i(x)R^{<i>})\right|_{v,Gauss}\right)\left|\mathrm{adj}\,R^{<0>}\right|_{v,Gauss}\left|\det R^{<0>}\right|^{-1}_{v,Gauss}\\ & \le & \left(\sup_{i\le s}\left|A_i(x)R^{<i>}\right|_{v,Gauss}\right)\left|\mathrm{adj}\,R^{<0>}\right|_{v,Gauss}\left|\det R^{<0>}\right|^{-1}_{v,Gauss}\\ & \le & C_{1,v}^s\left(\sup_{i\le s+\nu-1}|\vec{R}_i|_{v,Gauss}\right)\left(\sup_{i\le\nu-1}|\vec{R}_i|_{v,Gauss}\right)^{\nu-1}|\Delta(x)|^{-1}_{v,Gauss}\,,\end{array}$$

where we have set:

$$C_{1,v}=\sup(1,|A_1(x)|_{v,Gauss})$$

and

$$\Delta(x)=\det R^{<0>}(x)\,.$$

Taking into account our choice of N and n and (7.12.2), for all $i\le n+\nu-1$ we have:

$$\begin{array}{rcl}\left|\vec{R}_i\right|_{v,Gauss} & \le & |Q_i(x)|^{-1}_{v,Gauss}|Q_i(x)|_{v,Gauss}|g|_{v,Gauss}\left|(\vec{y})_{\le N+it}\right|_{v,Gauss}\\ & \le & |g|_{v,Gauss}\left|(\vec{y})_{\le N+it}\right|_{v,Gauss}\,,\end{array}$$

therefore:

$$\begin{array}{rcl}\sup_{s\le n}\log^+\left|G_{[}s]\right|_{v,Gauss} & \le & n\log C_{1,v}+\widetilde{h}\left(N+(n+\nu-1)t,v\right)\\ & & +(\nu-1)\widetilde{h}\left(N+(\nu-1)t,v\right)+\nu h(g,v)+\log^+|\Delta|^{-1}_{v,Gauss}\,.\end{array}$$

We set:

$$\begin{array}{rcl}\overline{\Delta}(x) & = & \vec{R}_0\wedge xQ_1(x)\vec{R}_1\wedge\cdots\wedge x^{\nu-1}Q_{\nu-1}(x)\vec{R}_{\nu-1}\\ & = & x^{\binom{\nu}{2}}\left(\prod_{i=1}^{\nu-1}Q_i(x)\right)\Delta(x)\,.\end{array}$$

The fact that $|Q_1(x)|_{v,Gauss}\le 1$ and $x^nQ^n(x)\vec{R}_n\in K[x]^\nu$, for all integers $n\ge 1$, implies that $\left|\overline{\Delta}(x)\right|_{v,Gauss}\le|\Delta(x)|_{v,Gauss}$, with $\overline{\Delta}(x)\in K[x]$, and:

$$\begin{array}{rcl}\sup_{s\le n}\log^+\left|G_{[}s]\right|_{v,Gauss} & \le & n\log C_{1,v}+\widetilde{h}\left(N+(n+\nu-1)t,v\right)\\ & & +(\nu-1)\widetilde{h}\left(N+(\nu-1)t,v\right)+\nu h(g,v)+\log^+\left|\overline{\Delta}\right|^{-1}_{v,Gauss}\,.\end{array}$$

Taking into account condition (7.12.1), we fix a positive integer k such that:

$$(7.12.3)\qquad \begin{cases} k > \dfrac{\nu(\nu-1)t}{1-\tau} \\ \dfrac{N}{n} = \dfrac{\nu t}{1-\tau} + \dfrac{k-\varepsilon_n}{n}\,, \text{ for some } \varepsilon_n \in (0,1) \text{ fixed.} \end{cases}$$

Let us set:

$$C_1 = \sum_{v\in\mathcal{P}_f} \log^+ |A_1(x)|_v$$

and

$$\Omega = \limsup_{n\to+\infty} \frac{1}{n}\left(\nu \sum_{v\in\mathcal{P}_f} h(g,v) + \sum_{v\in\mathcal{P}_f} \log^+ \left|\overline{\Delta}(x)\right|^{-1}_{v,Gauss}\right).$$

We obtain:

$$\begin{aligned}
\sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M}) &= \limsup_{n\to+\infty} \frac{1}{n}\left(\sum_{\substack{v\in\mathcal{P}_f \\ |1-q^\kappa|^{1/(p-1)}}} \sup_{s\le n} \log^+ \left|\frac{G_s}{[s]^!_q}\right|_{v,Gauss}\right) \\
&\le \sigma_f(\vec{y}) \limsup_{n\to+\infty}\left(\frac{N+(n+\nu-1)t}{n} + (\nu-1)\frac{N+(\nu-1)t}{n}\right) + C_1 + \Omega \\
&\le \sigma_f(\vec{y})\left(\frac{\nu t}{1-\tau} + t + (\nu-1)\frac{\nu t}{1-\tau}\right) + C_1 + \Omega \\
&\le \sigma_f(\vec{y})\left(\frac{\nu^2 t}{1-\tau} + t\right) + C_1 + \Omega\,.
\end{aligned}$$

□

7.5. **Step 4. Conclusion of the proof of Theorem 4.2.**

Lemma 7.13. *Let Ω be as in the previous proposition. Then:*

$$\Omega \le \frac{\nu^2 t}{1-\tau}\sigma_\infty(\vec{y}) + \frac{\nu^2 t(\nu-1)}{1-\tau} C_2 + \limsup_{n\to+\infty} \frac{\nu}{n} h(q)\,,$$

where

$$C_2 = \sum_{v\in\mathcal{P}_\infty} \log(1+|q|_v)$$

is a constant depending on the v-adic absolute value of q, for all $v \in \mathcal{P}_\infty$.

Proof. Let ξ a root of unity such that:

$$\overline{\Delta}(\xi) \ne 0 \ne Q_i(\xi)\ \forall\, i = 0,\dots \nu-1\,.$$

Since $|\overline{\Delta}(\xi)|_v \le |\overline{\Delta}(x)|_{v,Gauss}$ for all $v \in \mathcal{P}_f$, the Product Formula implies that:

$$\sum_{v\in\mathcal{P}_f} \log^+ \left|\overline{\Delta}(x)\right|^{-1}_{v,Gauss} \le \sum_{v\in\mathcal{P}_f} \log^+ \left|\overline{\Delta}(\xi)\right|^{-1}_{v} \le \sum_{v\in\mathcal{P}_\infty} \log^+ \left|\overline{\Delta}(\xi)\right|_{v}\,.$$

We recall that:

$$\overline{\Delta}(x) = \det\begin{pmatrix}\vec{R}_0 & Q_1(x)\vec{R}_1 & \cdots & Q_{\nu-1}(x)\vec{R}_{\nu-1}\end{pmatrix}$$

and that for all $s \le \nu-1$, (7.12.2) is verified. Moreover we have:

$$\begin{aligned}
Q_s(x)\frac{d_q^s(g)}{[s]^!_q}(x)\vec{y}(x) &= \sum_{n\ge0}\left(\sum_{i+j+h=n} (Q_s)_i \left(\frac{d_q^s(g)}{[s]^!_q}\right)_j \vec{y}_h\right) x^n \\
&= \sum_{n\ge0}\left(\sum_{i+j+h=n} (Q_s)_i \binom{s+j}{j}_q g_{s+j}\vec{y}_h\right) x^n\,,
\end{aligned}$$

where we have used the notation:

for all $P \in K[[x]]$ and for all $n \in \mathbb{N}$, P_n is the coefficient of x^n in P.

We deduce that $Q_s(\xi)\vec{R}_s(\xi)$ is a sum of terms of the type:

$$(Q_s)_i \binom{s+j}{j}_q g_{s+j}\vec{y}_h\xi^n$$

with:

$$\begin{array}{ll} 0\leq s\leq \nu-1\,, & 0\leq i\leq \deg Q_s(x)\,,\\ 0\leq j\leq N\,,\ s+j\leq N\,, & 0\leq h\leq N+(\nu-1)t\,. \end{array}$$

For all $v\in\mathcal{P}_\infty$ we obtain:

$$\left|Q_s(\xi)\vec{R}_s(\xi)\right|_v \leq c_v\left(\sup_{s\leq j\leq N}\left|\binom{j}{s}_q\right|_v\right)\left(\sup_{h\leq N+(\nu-1)t}|\vec{y}_h|_v\right)\left(\sup_{j\leq N}|g_j|_v\right)\,,$$

with:

$$c_v=\sup\left(1,\sup_{\substack{s=0,\dots,\nu-1\\ i=0,\dots,\deg Q_s}}|(Q_s(x))_i|_v\right)\,.$$

Since $|q|_v\neq 1$, for all $v\in\mathcal{P}_\infty$, we have:

$$\begin{array}{rcl} \left|\binom{j}{s}_q\right|_v & = & \left|\dfrac{(1-q^j)\cdots(1-q^{j-s+1})}{(1-q^s)\cdots(1-q)}\right|_v\\ & \leq & \dfrac{(1+|q|_v^j)\cdots(1+|q|_v^{j-s+1})}{|1-|q|_v^s|_v\cdots|1-|q|_v|_v}\\ & \leq & \begin{cases} \dfrac{(1+|q|_v)^s}{1-|q|_v^s}\leq\left(\dfrac{1+|q|_v}{1-|q|_v}\right)^{\nu-1} & \text{if } |q|_v<1;\\[2ex] \left(\dfrac{1+|q|_v^j}{|q|_v^s-1}\right)^s\leq\left(\dfrac{1+|q|_v^N}{|q|_v^{\nu-1}-1}\right)^{\nu-1} & \text{if } |q|_v>1; \end{cases} \end{array}$$

hence:

$$\begin{array}{rcl} \sup\limits_{\substack{s=0,\dots,\nu-1\\ j=s,\dots N}}\left|\binom{j}{s}_q\right|_v & \leq & \left(\dfrac{\sup(1+|q|_v,1+|q|_v^N)}{\inf(|1-|q|_v|,|1-|q|_v^{\nu-1}|)}\right)^{\nu-1}\\ & \leq & \dfrac{(1+|q|_v)^{N(\nu-1)}}{\inf(|1-|q|_v|,|1-|q|_v^{\nu-1}|)^{\nu-1}} \end{array}$$

We obtain the following estimate:

$$\left|Q_s(\xi)\vec{R}_s(\xi)\right|_v\leq c_v\frac{(1+|q|_v)^{N(\nu-1)}}{\inf(|1-|q|_v|,|1-|q|_v^{\nu-1}|)^{\nu-1}}\left(\sup_{h\leq N+(\nu-1)t}|\vec{y}_h|_v\right)\left(\sup_{j\leq N}|g_j|_v\right)\,.$$

Finally we get:

$$\left|\overline{\Delta}(\xi)\right|_v\leq c_v^\nu\frac{(1+|q|_v)^{N(\nu-1)\nu}}{\inf(|1-|q|_v|,|1-|q|_v^{\nu-1}|)^{(\nu-1)\nu}}\left(\sup_{h\leq N+(\nu-1)t}|\vec{y}_h|_v\right)^\nu\left(\sup_{j\leq N}|g_j|_v\right)^\nu$$

and therefore:

$$\begin{array}{l} \displaystyle\sum_{v\in\mathcal{P}_\infty}\log^+\left|\overline{\Delta}(\xi)\right|_v\leq const+N\nu(\nu-1)C_2\\ \qquad\displaystyle -\nu(\nu-1)\sum_{v\in\mathcal{P}_\infty}\log\inf(|1-|q|_v|,|1-|q|_v^{\nu-1}|)^{\nu-1}\\ \qquad\displaystyle +\nu\sum_{v\in\mathcal{P}_\infty}h(g,v)+\nu\sum_{v\in\mathcal{P}_\infty}\widetilde{h}\left(N+(\nu-1)t,v\right)\,. \end{array}$$

where:

$$C_2=\sum_{v\in\mathcal{P}_\infty}\log\left(1+|q|_v\right)\,.$$

We recall that by (7.12.3), we have:

$$\lim_{n\to+\infty}\frac{N}{n}=\frac{t\nu}{1-\tau}$$

and:

$$\lim_{n\to+\infty}\frac{\log N}{n}=0\,.$$

So we can conclude since:

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\sum_{v\in\mathcal{P}_f}\log^+\left|\overline{\Delta}(x)\right|_{v,Gauss}^{-1}&\leq\limsup_{n\to+\infty}\frac{1}{n}\sum_{v\in\mathcal{P}_\infty}\log^+\left|\overline{\Delta}(\xi)\right|_v\\ &\leq \limsup_{n\to+\infty}\left(\frac{N\nu(\nu-1)C_2}{n}+\frac{\nu}{n}\sum_{v\in\mathcal{P}_\infty}h(g,v)+\frac{\nu}{n}\sum_{v\in\mathcal{P}_\infty}\widetilde{h}\left(N+(\nu-1)t,v\right)\right)\\ &\leq \frac{t\nu^2(\nu-1)}{1-\tau}C_2+\limsup_{n\to+\infty}\left(\frac{\nu}{n}\sum_{v\in\mathcal{P}_\infty}h(g,v)+\frac{\nu}{n}\sum_{v\in\mathcal{P}_\infty}\widetilde{h}\left(N+(\nu-1)(t-1),v\right)\right)\\ &\leq \frac{t\nu^2}{1-\tau}\sigma_\infty(\vec{y})+\frac{t\nu^2(\nu-1)}{1-\tau}C_2+\limsup_{n\to+\infty}\frac{\nu}{n}\sum_{v\in\mathcal{P}_\infty}h(g,v)\,.\end{aligned}$$

□

Conclusion of the proof of Theorem 4.2. Proposition 7.4 implies that:

$$\begin{aligned}\limsup_{n\to+\infty}\frac{\nu}{n}h(g)&\leq\limsup_{n\to+\infty}\frac{\nu}{n}\left(const+\frac{1-\tau}{\tau}\sum_{v\in\mathcal{P}}\widetilde{h}\left(N+N\frac{1-\tau}{\nu},v\right)\right)\\ &\leq \limsup_{n\to+\infty}\frac{1-\tau}{\tau}\frac{\nu}{n}\sum_{v\in\mathcal{P}}\widetilde{h}\left(N+N\frac{1-\tau}{\nu},v\right)\\ &\leq \frac{1-\tau}{\tau}\nu\sigma\left(\vec{y}\right)\limsup_{n\to+\infty}\frac{1}{n}\left(N+N\frac{1-\tau}{\nu}\right)\\ &\leq \frac{1-\tau}{\tau}\nu\sigma\left(\vec{y}\right)\left(\frac{t\nu}{1-\tau}+t\right)\\ &\leq \frac{1-\tau}{\tau}\nu t\left(1+\frac{\nu}{1-\tau}\right)\sigma\left(\vec{y}\right)\,,\end{aligned}$$

which, combined with Propositions 7.12 and 7.13, implies that:

$$\begin{aligned}\sigma_{\mathcal{P}_f}^{(q)}(\mathcal{M})&\leq\sigma_f(\vec{y})\left(\frac{\nu^2t}{1-\tau}+t\right)+\sigma_\infty\left(\vec{y}\right)\frac{\nu^2t}{1-\tau}+\sigma\left(\vec{y}\right)\frac{1-\tau}{\tau}\nu t\left(1+\frac{\nu}{1-\tau}\right)\\ &\quad+\log C_1+\frac{\nu^2(\nu-1)t}{1-\tau}C_2\\ &\leq \sigma(\vec{y})\left(\frac{\nu^2t}{1-\tau}+\nu^2t\left(\frac{1}{\tau}+\frac{1-\tau}{\nu\tau}\right)+t\right)+\log C_1+\frac{\nu^2(\nu-1)t}{1-\tau}C_2\\ &\leq \sigma(\vec{y})\left(\nu^2t\left(\frac{\nu+1}{\nu}\frac{1}{\tau}+\frac{1}{1-\tau}\right)-\nu t+t\right)+\log C_1+\frac{\nu^2(\nu-1)t}{1-\tau}C_2\,.\end{aligned}$$

The function $\frac{\nu+1}{\nu}\frac{1}{\tau}+\frac{1}{1-\tau}$ has a minimum for

$$\tau=\left(1+\sqrt{\frac{\nu}{\nu+1}}\right)^{-1}\,;$$

for this value of τ we get:

$$\frac{\nu+1}{\nu}\frac{1}{\tau}+\frac{1}{1-\tau}=\left(1+\sqrt{\frac{\nu+1}{\nu}}\right)\leq\begin{cases}4.95 & \text{for } \nu\geq 2\\ 5.9 & \text{for } \nu=1\end{cases}\,.$$

Finally we have:

$$\sigma_{\mathcal{P}_f}^{(q)}(\mathcal{M})\leq\log C_1+\frac{\nu^2(\nu-1)t}{1-\tau}C_2+\begin{cases}\sigma(\vec{y})\left(4.95\nu^2t-\nu t+(t-1)\right) & \text{for } \nu\geq 2\\ \\ \sigma(\vec{y})5.9t & \text{for } \nu=1\end{cases}\,,$$

where

$$C_1=\sum_{v\in\mathcal{P}_f}\log^+|A_1(x)|_{v,Gauss}$$

and

$$C_2 = \sum_{v\in\mathcal{P}_\infty} \log(1+|q|_v)\,.$$

□

Part 2. Global q-Gevrey series

8. Definition and first properties

The notation is the same as in Part 1. We recall that K is a finite extension of $k(q)$, equipped with its family of ultrametric norms, normalized so that the Product Formula holds. The field $K(x)$ is naturally a q-difference algebra with respect to the operator $\sigma_q : f(x) \mapsto f(qx)$.

Definition 8.1. We say that the series $f(x)=\sum_{n=0}^\infty a_n x^n \in K[[x]]$ is a *global q-Gevrey series of orders* $(s_1,s_2)\in\mathbb{Q}^2$ if it is solution of a q-difference equation with coefficients in $K(x)$ and

$$\sum_{n=0}^\infty \frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]_q^!\right)^{s_2}} x^n$$

is a G_q-function.

Remark 8.2. We point out that:

(1) The definition above forces s_2 to be an integer, in fact the q-holonomy condition implies that the coefficients $[n]_q^{!\,s_2}$, for $n\geq 1$, are all contained in a finite extension of $k(q)$.

(2) Being a global q-Gevrey series of orders (s_1,s_2) implies being a q-Gevrey series of order s_1+s_2 in the sense of [BB92] for all $v\in\mathcal{P}_\infty$ extending the q^{-1}-adic norm, *i.e.* for the norms that verify $|q|_v>1$: this simply means that $|q^{\frac{s_1 n(n-1)}{2}}[n]_q^{!\,s_2}|_v$ as the same growth as $|q|_v^{(s_1+s_2)\frac{n(n-1)}{2}}$. If $v\in\mathcal{P}_\infty$ and $|q|_v<1$, then $|[n]_q|_v=1$. Therefore a global q-Gevrey series of orders (s_1,s_2) is a q-Gevrey series of order s_1 in the sense of [BB92]. This remark actually justifies the the choice of considering two orders, instead of one as in the analytic theory.

In the local case, both complex (*cf.* [Béz92], [MZ00], [Zha99]) and p-adic (*cf.* [BB92]), the q-Gevrey order is not uniquely determined. The global situation considered here is much more rigid: the same happens in the differential case.

Proposition 8.3. *The orders of a given global q-Gevrey series* $\sum_{n=0}^\infty a_n x^n \in K[[x]]\smallsetminus K[x]$ *are uniquely determined.*

Proof. Suppose that $\sum_{n=0}^\infty a_n x^n$ is a global q-Gevrey series of orders (s_1,s_2) and (t_1,t_s). By definition

$$\sum_{n=0}^\infty \frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]_q^!\right)^{s_2}} x^n \text{ and } \sum_{n=0}^\infty \frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{t_1}\left([n]_q^!\right)^{t_2}} x^n$$

have finite size. We have:

$$\sum_{n=0}^\infty \frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{t_1}\left([n]_q^!\right)^{t_2}} x^n = \sum_{n=0}^\infty \left(q^{\frac{n(n-1)}{2}}\right)^{s_1-t_1}\left([n]_q^!\right)^{s_2-t_2}\frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]_q^!\right)^{s_2}} x^n\,.$$

One observes that having finite size implies having finite radius of convergence for all $v\in\mathcal{P}$, therefore for all v such that $|q|_v\neq 1$ we must have:

$$\limsup_{n\to\infty}\left|\left(q^{\frac{n(n-1)}{2}}\right)^{s_1-t_1}\left([n]_q^!\right)^{s_2-t_2}\right|_v^{1/n}<\infty\,.$$

If $|q|_v>1$ this implies:

$$\limsup_{n\to\infty}|q|_v^{\frac{n-1}{2}(s_1+s_2-(t_1+t_2))}<\infty\,.$$

Since for all $v\in\mathcal{P}$ such that $|q|_v<1$ the limit $\limsup_{n\to\infty}|[n]_q^!|_v^{1/n}$ is bounded we get:

$$\limsup_{n\to\infty}|q|_v^{\frac{n-1}{2}(s_1-t_1)}<\infty\,.$$

We deduce that necessarily $s_1+s_2 \leq t_1+t_2$ and $t_1 \leq s_1$, hence $t_1 \leq s_1$ and $s_2 \leq t_2$. Since the role of (t_1,t_2) and (s_1,s_2) is symmetric, one obviously obtain the opposite inequalities in the same way. □

8.1. **Changing q in q^{-1}.** One can transform a q-difference equations in a q^{-1}-difference equations, obtaining:

Proposition 8.4. *Let $f(x)\in K[[x]]$ be a global q-Gevrey series of orders $(-s_1,-s_2)\in\mathbb{Q}^2$, then $f(x)$ is a global q^{-1}-Gevrey series of orders $(s_1+s_2,-s_2)$.*

In particular, if $f(x)$ is a global q-Gevrey series of orders $(t_1,-t_2)$, with $t_1\geq t_2\geq 0$, then $f(x)$ is a global q^{-1}-Gevrey series of negative orders $(-(t_1-t_2),-t_2)$.

Proof. It is enough to write $f(x)$ in the form:

$$f(x)=\sum_{n=0}^{\infty}\frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]_q^!\right)^{s_2}}x^n=\sum_{n=0}^{\infty}\frac{a_n}{\left(q^{-\frac{n(n-1)}{2}}\right)^{-s_1-s_2}\left([n]_{q^{-1}}^!\right)^{s_2}}x^n\,,$$

where $\sum_n a_nx^n$ is a convenient G_q-function. □

8.2. **Rescaling of the orders.** Clearly we can always look at a global q-Gevrey series of orders $(s,0)$ as a global q^t-Gevrey series of orders $(s/t,0)$, for any $t\in\mathbb{Q}$, $t\neq 0$, the holonomy condition being always satisfied:

Lemma 8.5. *Let $t\in\mathbb{Q}$, $t\neq 0$. If $f(x)=\sum_{n=0}^{\infty}a_nx^n$ is solution of a q-difference equation then it is solution of a q^t-difference equation.*

Proof. If $f(x)$ is solution of a q-difference equation, then it is also solution of a q^{-1}-difference equation. Therefore we can suppose $t>0$. Let $t=\frac{p}{r}$, with $p,r\in\mathbb{Z}_{>0}$. Since $f(x)$ is solution of a q-difference operator, we have:

$$\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_q^i\left(f(x)\right)<+\infty\,.$$

Then:

$$\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_{q^p}^i\left(f(x)\right)=\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_q^{ip}\left(f(x)\right)\leq\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_q^i\left(f(x)\right)<+\infty\,,$$

so $f(x)$ is solution of a q^p-difference operator. Finally we can conclude since $\sum_{i=0}^{\nu}a_i(x)f(q^{pi}x)=0$ implies that $\sum_{i=0}^{\nu}a_i(x)f(\widetilde{q}^{tir}x)=0$. □

Unfortunately, the same is not true for global q-Gevrey series of orders $(0,s)$. To prove it, one can calculate size of the series

$$\Phi(x)=\sum_{n\geq 0}\frac{(\widetilde{q};\widetilde{q})_n^t}{(q;q)_n}x^n\,,$$

where $\widetilde{q}$ is a r-th root of q, for some positive integer r, $K=\mathbb{Q}(\widetilde{q})$ and t is an integer. The Pochhammer symbols $(\widetilde{q};\widetilde{q})_n^t$ and $(q;q)_n$ are both polynomials in $\widetilde{q}^{1/2}$ of degree $tn(n+1)$ and $rn(n+1)$, respectively. If we want $\Phi(x)$ to have finite size, we are forced to take $t\leq r$, so that it has positive radius of convergence at any place v such that $|q|_v>1$. Notice that $\Phi(x)$ is convergent for any place v such that $|q|_v<1$ and that the noncyclotomic places give a zero contribution to the size. As far as the cyclotomic places of K is concerned, we obtain

$$\sigma_{\mathcal{C}}(y)=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{rn}\left(\left[\frac{n}{k}(k,r)\right]-t\left[\frac{n}{k}\right]\right)\log^+d^{-1}\sim\limsup_{n\to\infty}\sum_{k=1}^{rn}\frac{1}{k}((k,r)-t)\log d^{-1}\,.$$

The limit above is infinite.

9. Formal Fourier transformations

The following natural two q-analogues of the usual formal Borel transformation

$$\begin{array}{rccc} (\cdot)^+ : & K[[x]] & \longrightarrow & K[[z^{-1}]] \\ & F=\sum_{n=0}^{\infty} a_n x^n & \longmapsto & F^+=\sum_{n=0}^{\infty}[n]_q^! a_n z^{-n-1} \end{array}$$

and

$$\begin{array}{rccc} (\cdot)^\# : & K[[x]] & \longrightarrow & K[[z^{-1}]] \\ & F=\sum_{n=0}^{\infty} a_n x^n & \longmapsto & F^\#=\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} a_n z^{-n-1} \end{array},$$

are equally considered in the literature on q-difference equations. From an archimedean analytical point of view, they are equivalent as soon as one works under the hypothesis that $|q|\neq 1$ (*cf.* [MZ00, §8] and [DVZ07, Part II]). As already noticed in [And00b], from a global point of view, $(\cdot)^+$ and $(\cdot)^\#$ have a completely different behavior: for the same reason the definition of global q-Gevrey series involves two orders.

Let $p=q^{-1}$ and let $\sigma_p : z\mapsto pz$, $d_p=\frac{\sigma_p-1}{(p-1)z}$[2]. The Borel transformations that we have introduced above have the following properties:

Lemma 9.1. *For all $F=\sum_{n=0}^{\infty} a_n x^n\in K[[x]]$ we have:*

$$\begin{array}{ll} (xF)^+ = -pd_pF^+\,, & (d_qF)^+ = zF^+ - F(0)\,, \\ (xF)^\# = \dfrac{p}{z}\sigma_pF^\#\,, & (\sigma_qF)^\# = p\sigma_pF^\#\,. \end{array}$$

Proof. We deduce the first equality using the relation:

$$-pd_p\frac{1}{z^n} = [n]_q\frac{1}{z^{n+1}}\,.$$

All the other formulas easily follow from the definitions. □

For any q-difference operator $\sum_{i=0}^N a_i(x)\sigma_q^i\in K(x)[\sigma_q]$ (resp. $\sum_{i=0}^N b_i(x)d_q^i\in K(x)[d_q]$) we set:

$$\deg_{\sigma_q}\sum_{i=0}^{\nu} a_i(x)\sigma_q^i = \sup\{i\in\mathbb{Z} : 0<i<\nu,\ a_i(x)\neq 0\}$$

$$\left(\text{resp. } \deg_{d_q}\sum_{i=0}^{\nu} b_i(x)d_q^i = \sup\{i\in\mathbb{Z} : 0<i<\nu,\ b_i(x)\neq 0\}\right)$$

Obviously we have $K(x)[d_q]=K(x)[\sigma_q]$ and $\deg_{d_q}=\deg_{\sigma_q}$ (for explicit formulas *cf.* [DV02, 1.1.10] and (10.0.3) below). The previous lemma justifies the definition of the formal Fourier transformations below, acting on the skew rings $K[x,d_q]$ and $K[x,\sigma_q]$:

Definition 9.2. We call the maps:

$$\begin{array}{llll} \mathcal{F}_{q^+} & : K[x,d_q] & \longrightarrow & K[z,d_p] \\ & d_q & \longmapsto & z \\ & x & \longmapsto & -pd_p \end{array} \quad\text{and}\quad \begin{array}{llll} \mathcal{F}_{q^\#} & : K[x,\sigma_q] & \longrightarrow & K\left[\frac{1}{z},\sigma_p\right] \\ & \sigma_q & \longmapsto & p\sigma_p \\ & x & \longmapsto & \frac{1}{qz}\sigma_p \end{array}$$

the q^+*-Fourier transformation* and the $q^\#$*-Fourier transformation* respectively.

[2] This notation is a little bit ambiguous and we should rather write $\sigma_{p,z}$, $d_{p,z}$, $d_{q,x}$, etc. etc. Anyway the contest will be always clear enough not to be obliged to specify the variable in the notation.

Remark 9.3. Let $\mathcal{F}_p : K[z, d_p] \to K[x, d_q]$ and let $\lambda : K[x, d_q] \to K[x, d_q]$, $d_q \mapsto -\frac{1}{q}d_q$, $x \mapsto -qx$ Then $\mathcal{F}_{q^+}^{-1} = \lambda \circ \mathcal{F}_{p^+}$.

As far as $\mathcal{F}_{q^\#}$ is concerned, if $\mathcal{L} = \sum_{i=0}^{\nu} a_i(\frac{1}{z})\sigma_p^i \in K\left[\frac{1}{z}, \sigma_p\right]$ is such that $\deg_{\frac{1}{z}} a_i\left(\frac{1}{z}\right) \leq i$, there exists a unique $\mathcal{N} \in K[x, \sigma_q]$ such that $\mathcal{F}_{q^\#}(\mathcal{N}) = \mathcal{L}$ and we note $\mathcal{F}_{q^\#}^{-1}(\mathcal{L}) = \mathcal{N}$.

In the following lemma we verify that the formal Fourier transformations we have just defined are compatible with the Borel transformations $(\cdot)^+$ and $(\cdot)^\#$:

Lemma 9.4. *Let $F \in K[[x]]$ be a series solution of a q-difference linear operator $\mathcal{N} \in K[x, d_q]$, such that $\nu = \deg_{d_q} \mathcal{N}$ (resp. $\mathcal{N} \in K[x, \sigma_q]$). Then $d_{q^{-1}}^{\nu} \circ \mathcal{F}_{q^+}(\mathcal{N})F^+ = 0$ (resp. $\mathcal{F}_{q^\#}(\mathcal{N})F^\# = 0$).*

Inversely:

1. *If F^+ is a solution of $\mathcal{L}_1 \in K[z, d_p]$, then $\mathcal{F}_{q^+}^{-1}(\mathcal{L}_1)F = 0$.*
2. *If $\mathcal{L}_2 \in K\left[\frac{1}{z}, \sigma_p\right]$ is such that $\mathcal{L}_2 F^\# = 0$, for all $n \in \mathbb{N}$, $n >> 0$, we have: $\mathcal{F}_{q^\#}^{-1}(\sigma_p^n \circ \mathcal{L}_2)F = 0$.*

Proof. We prove the statements for $(\cdot)^+$. The proof for $(\cdot)^\#$ is quite similar. We write $\mathcal{N}$ in the form:

$$\mathcal{N} = \sum_{j=0}^{\nu} \sum_{i=0}^{N} a_{i,j} x^i d_q^j \in K[x, d_q] .$$

Lemma 9.1 implies that $\mathcal{F}_{q^+}(\mathcal{N})F^+$ is a polynomial of degree less or equal to ν, therefore $d_{q^{-1}}^{\nu} \circ \mathcal{F}_{q^+}(\mathcal{N})F^+ = 0$. Let us now write $\mathcal{L}_1$ as:

$$\mathcal{L}_1 = \sum_{j=0}^{\nu} \sum_{i=0}^{N} a_{i,j} z^i d_p^j \in K[z, d_p] .$$

Then $\left(\mathcal{F}_{q^+}^{-1}(\mathcal{L}_1)F\right)^+$ is a polynomial of degree less or equal to ν. Hence we obtain:

$$d_p^{\nu}\left(\mathcal{F}_{q^+}^{-1}(\mathcal{L}_1)F\right)^+ = \left((-qx)^{\nu}\mathcal{F}_{q^+}^{-1}(\mathcal{L}_1)F\right)^+ = 0$$

and finally $(-qx)^{\nu}\mathcal{F}_{q^+}^{-1}(\mathcal{L}_1)F = 0$. □

Remark 9.5. In the following we will use the formal Fourier transformations above composed with the symmetry $S : z \mapsto 1/x$:

$$\begin{array}{cccccccccc} S \circ \mathcal{F}_{q^+} & : K[x, d_q] & \longrightarrow & K\left[\frac{1}{x}, x, d_q\right] & \text{and} & S \circ \mathcal{F}_{q^\#} & : K[x, \sigma_q] & \longrightarrow & K[x, \sigma_q] \\ & d_q & \longmapsto & \frac{1}{x} & & & \sigma_q & \longmapsto & \frac{1}{q}\sigma_q \\ & x & \longmapsto & x^2 d_q & & & x & \longmapsto & \frac{x}{q}\sigma_q \end{array} \quad (9.5.1)$$

Notice that $S \circ \mathcal{F}_{q^+}(d_q \circ x) = x d_q$.

10. Action of the formal Fourier transformations on the Newton Polygon

Let as consider a linear q-difference operator:

$$\mathcal{N} = \sum_{i=0}^{\nu} a_i(x) x^i d_q^i = \sum_{i=0}^{\nu} b_i(x)\sigma_q^i \, , \quad (10.0.2)$$

such that $b_j(x), a_j(x) \in K[x]$. Applying formulas [DV02, 1.1.10], we obtain:

$$\begin{aligned} \mathcal{N} &= \sum_{j=0}^{\nu} b_j(x) \sum_{i=0}^{j} \binom{j}{i}_q (1-q)^i q^{i(i-1)/2} x^i d_q^i \\ &= \sum_{i=0}^{\nu} (1-q)^i q^{i(i-1)/2} \left(\sum_{j=i}^{\nu} \binom{j}{i}_q b_j(x) \right) x^i d_q^i \, . \end{aligned} \quad (10.0.3)$$

Therefore $a_i(x) = (1-q)^i q^{i(i-1)/2} \sum_{j=i}^{\nu} \binom{j}{i}_q b_j(x)$.

We recall the definition of the Newton-Ramis Polygon:

Definition 10.1. Let $\mathcal{N}=\sum_{i=0}^{\nu}a_i(x)x^i d_q^i=\sum_{i=0}^{\nu}b_i(x)\sigma_q^i$ be such that $b_j(x),a_j(x)\in K[x]$. Then we define *the Newton-Ramis Polygon of $\mathcal{N}$ with respect to σ_q* (and we write $NRP_{\sigma_q}(\mathcal{N})$) (resp. *with respect to d_q* (and we write $NRP_{d_q}(\mathcal{N})$)) to be the convex hull of the following set:

$$\bigcup_{b_i(x)\neq 0}\{(u,v)\in\mathbb{R}^2 : u=i, \deg_x b_i(x)\geq v\geq \mathrm{ord}_x b_i(x)\}\subset\mathbb{R}^2\,.$$

$$\left(\text{resp.}\quad \bigcup_{a_i(x)\neq 0}\{(u,v)\in\mathbb{R}^2 : u\leq i, \deg_x a_i(x)\geq v\geq \mathrm{ord}_x a_i(x)\}\subset\mathbb{R}^2\right)\,.$$

For an operator with rational coefficient $\mathcal{N}$, we set $NRP_{\sigma_q}(\mathcal{N})=NRP_{\sigma_q}(f(x)\mathcal{N})$ and $NRP_{d_q}(\mathcal{N})=NRP_{d_q}(f(x)\mathcal{N})$, where $f(x)$ is a polynomial in $K[x]$ such that $f(x)\mathcal{N}$ can be written as above. In this way the Newton-Ramis polygon is defined up to a vertical shift, so that its slopes are actually well-defined.

Lemma 10.2. *We have:*

$$NRP_{d_q}(\mathcal{N})=\bigcup_{(u_0,v_0)\in NRP_{\sigma_q}(\mathcal{N})}\{(u,v_0)\in\mathbb{R}^2 : u\leq u_0\}\,.$$

Proof. The statement follows from (10.0.3). □

The following proposition describes the behavior of the Newton-Ramis Polygon with respect to $\mathcal{F}_{q^+}$ and $\mathcal{F}_{q^\#}$.

Proposition 10.3. *The map*[3]*:*

$$\begin{array}{ccc} NRP_{\sigma_q}(\mathcal{N}) & \longrightarrow & NRP_{\sigma_p}\left(\mathcal{F}_{q^\#}(\mathcal{N})\right)\\ (u,v) & \longmapsto & (u+v,-v)\end{array}\qquad \left(\textit{resp.}\ \begin{array}{ccc} NRP_{d_q}(\mathcal{N}) & \longrightarrow & NRP_{d_p}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)\\ (u,v) & \longmapsto & (u+v,-v)\end{array}\right)$$

is a bijection between $NRP_{\sigma_q}(\mathcal{N})$ and $NRP_{\sigma_p}\left(\mathcal{F}_{q^\#}(\mathcal{N})\right)$ (resp. $NRP_{d_q}(\mathcal{N})$ and $NRP_{d_p}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)$).

Proof. As far as $\mathcal{F}_{q^\#}$ is concerned, it is enough to notice that:

$$\mathcal{F}_{q^\#}\left(\sum_{i=0}^{\nu}\sum_{j=0}^{N}b_{i,j}x^j\sigma_q^i\right)=\sum_{i=0}^{\nu}\sum_{j=0}^{N}\frac{b_{i,j}}{q^{j(j-3)/2}q^i}\frac{1}{z^j}\sigma_p^{i+j}\,.$$

Let

$$\mathcal{N}=\sum_{i=0}^{\nu}\sum_{j=0}^{N}a_{i,j}x^j d_q^i\,.$$

We have:

$$\begin{aligned}\mathcal{F}_{q^+}(\mathcal{N}) &= \sum_{i=0}^{\nu}\sum_{j=0}^{N}\frac{(-1)^j a_{i,j}}{q^j}d_p^j\circ z^i\\ &= \sum_{i=0}^{\nu}\sum_{j=0}^{\nu}\sum_{h=0}^{j}\frac{(-1)^j a_{i,j}}{q^j}\binom{j}{h}_q\frac{[i]_q^!}{[h-i]_q^!}q^{(j-h)(i-h)}z^{i-h}d_p^{j-h}\,.\end{aligned}$$

Then if $(i,j-i)\in NRP_{d_q}(\mathcal{N})$ we have:

$$(j-h,i-j)\in NRP_{d_p}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)\ \text{ for all } h=0,\ldots,j.$$

The statement follows from this remark. □

By convention, the vertical sides of $NRP_{\sigma_q}(\mathcal{N})$) (resp. $NRP_{d_q}(\mathcal{N})$) have slope ∞. The opposite of the finite slopes of the "upper part" of $NRP_{\sigma_q}(\mathcal{N})$ are the slopes at ∞ of $\mathcal{N}$ while the finite slopes of the "lower part" are the slopes of $\mathcal{N}$ at 0.

[3]To make the notation clear, we underline that we denote $NRP_{\sigma_p}\left(\mathcal{F}_{q^\#}(\mathcal{N})\right)$ the Newton-Ramis Polygon of $\mathcal{F}_{q^\#}(\mathcal{N})$ defined with respect to z and σ_p and $NRP_{d_p}\left(\mathcal{F}_{q^+}(\mathcal{N})\right)$ the Newton-Ramis Polygon of $\mathcal{F}_{q^+}(\mathcal{N})$ defined with respect to z and d_p.

Corollary 10.4. *In the notation of the previous proposition, $\mathcal{F}_{q^\#}$ (resp. $\mathcal{F}_{q^+}$) acts in the following way on the slopes of the Newton-Ramis Polygon:*

$$\begin{array}{cccc} & \{\text{slopes of } NRP_{\sigma_q}(\mathcal{N})\} & \longrightarrow & \{\text{slopes of } NRP_{\sigma_p}\left(\mathcal{F}_{q^\#}(\mathcal{N})\right)\} \\ \Big(\text{resp.} & \{\text{slopes of } NRP_{d_q}(\mathcal{N})\} & \longrightarrow & \{\text{slopes of } NRP_{d_p}\left(\mathcal{F}_{q^\#}(\mathcal{N})\right)\} \quad \Big). \\ & \lambda & \longmapsto & -\frac{\lambda}{1+\lambda} \\ & \infty & \longmapsto & -1 \end{array}$$

11. Solutions at points of K^*

We have described what happens at zero and at ∞ when the Fourier transformations act. Now we want to describe what happens at a point $\xi \in K^* = \mathbb{P}^1(K) \smallsetminus \{0, \infty\}$.

To construct some formal solutions of our q-difference operators at $\xi \in K^*$, we are going to consider a ring defined as follows (*cf.* [DV04, §1.3]). For any $\xi \in K$ and any nonnegative integer n, we consider the polynomials

$$T_n^q(x,\xi) = x^n \left(\frac{\xi}{x}; q\right)_n = (x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi)\,.$$

One verifies directly that for any $n \geq 1$

$$d_q T_n^q(x,\xi) = [n]_q T_{n-1}^q(x,\xi)$$

and $d_q T_0^q(x,\xi) = 0$. The product $T_n^q(x,\xi) T_m^q(x,\xi)$ can be written as a linear combination with coefficients in K of $T_0^q(x,\xi), T_1^q(x,\xi), \ldots, T_{n+m}^q(x,\xi)$ (*cf.* [DV04, §1.3]). It follows that we can define the ring:

$$K[[x-\xi]]_q = \left\{ \sum_{n\geq 0} a_n T_n^q(x,\xi) : a_n \in K \right\},$$

with the obvious sum and the Cauchy product described above, extended by linearity. The ring $K[[x-\xi]]_q$ is a q-difference algebra with the natural action of d_q. Notice that in general it makes no sense to look at the sum of those series. Nevertheless, they can be evaluated at the point of the set $\xi q^{\mathbb{Z}_{\geq 0}}$, and they are actually in bijective correspondence with the sequences $\{f(\xi q^n)\}_{n\in\mathbb{Z}_{\geq 0}} \in \mathbb{C}^{\mathbb{N}}$.

Proposition 11.1. *Let $\mathcal{N} \in K[x, d_q]$ be a linear q-difference operator such that $NRP_{d_q}(\mathcal{N})$ has only the zero slope at ∞[4]; then the operator $\mathcal{F}_{q^+}\mathcal{N}$ has a basis of solution in $K[[z-\xi]]_p$ for all $\xi \in K^*$.*

Proof. The hypothesis on the Newton Polygon of $\mathcal{N}$ at ∞ implies that we can write $\mathcal{N}$ in the following form

$$\mathcal{N} = \sum_{i=0}^{\nu}\sum_{j=0}^{N} a_{i,j} x^j d_q^i\,,$$

with $a_{i,N} = 0$ for all $i = 0, \ldots, \nu - 1$ and $a_{\nu,N} \neq 0$. This implies that the coefficient of d_p^N in

$$\begin{aligned} \mathcal{F}_{q^+}(\mathcal{N}) &= \sum_{i=0}^{\nu}\sum_{j=0}^{N} a_{i,j}\left(-p d_p\right)^j \circ z^i \\ &= \sum_{j=0}^{N-1}\sum_{i=0}^{\nu} c_{j,i} z^i d_p^j + a_{\nu,N}\left(-q\right)^{\nu-N} z^\nu d_p^N \end{aligned}$$

does not have any zero in the set $\{q^n\xi : n \in \mathbb{Z}_{>0}\}$. Using the fact that $d_p T_n^p(z,\xi) = [n]_p T_{n-1}^p(z,\xi)$ and that $z T_n^p(z,\xi) = T_{n+1}^p(z,\xi) + p^n \xi T_n^p(z,\xi)$, a basis of solutions of $\mathcal{F}_{q^+}(\mathcal{L})$ in $K[[z-\xi]]_p$ can be constructed working with the recursive relation induced by $\mathcal{F}_{q^+}(\mathcal{L})y = 0$ on the coefficients of a generic solution of the form $\sum_n \alpha_n T_n^p(z,\xi)$. □

[4] or equivalently, $NRP_{d_q}(\mathcal{N})$ has no negative slopes.

Corollary 11.2. *For any $\mathcal{N}\in K[z,d_p]$ (resp. $\mathcal{N}\in K[x,d_q]$, $\mathcal{N}\in K[z,d_p]$) having only the zero slope at ∞, the operator $\mathcal{F}_{q^+}^{-1}(\mathcal{N})$ (resp. $S\circ\mathcal{F}_{q^+}(\mathcal{N})$, $S\circ\mathcal{F}_{q^+}^{-1}(\mathcal{N})$) has a basis of solution in $K\left[[x-\xi]\right]_q$ for any $\xi\in K^*$.*

Proof. The statement follows from the remark that $\mathcal{F}_{q^+}^{-1}(\mathcal{N})=\lambda\circ\mathcal{F}_{p^+}(\mathcal{N})$ and that the symmetry $S:z\mapsto 1/x$ does not changes the kind of singularity at the points of K^*. □

An analogous property holds for $\mathcal{F}_{q^\#}^{-1}$:

Proposition 11.3. *Let $\mathcal{L}=\sum_{i=0}^{\nu}a_i\left(\frac{1}{z}\right)\sigma_p^i\in K\left[\frac{1}{z},\sigma_p\right]$ such that $\deg_{\frac{1}{z}}a_i(\frac{1}{z})\leq i$. We suppose that*

$$N=\mathrm{ord}_{\frac{1}{z}}a_\nu\left(\frac{1}{z}\right)\leq\mathrm{ord}_{\frac{1}{z}}a_i\left(\frac{1}{z}\right),$$

for all $i=0,\dots,\nu-1$[5]. Then $\mathcal{F}_{q^\#}^{-1}(\mathcal{L})$ has a basis of solution in $K[[x-\xi]]_q$ for all $\xi\in K^*$.

Proof. We call $a_{\nu,N}\in K$ the coefficients of $\frac{1}{z}^N$ in $a_\nu\left(\frac{1}{z}\right)$. Then we have:

$$\mathcal{F}_{q^\#}^{-1}(\mathcal{L})=\sum_{i=0}^{\nu-N-1}b_i(x)\sigma_q^i+a_{\nu,N}x^N\sigma_q^{\nu-N}.$$

One ends the proof as above. □

12. Structure theorems

Inspired by [And00a], we want to characterize q-difference operators killing a global q-Gevrey series of orders $(-s_1,-s_2)$, with $(s_1,s_2)\in\mathcal{Z}:=\mathbb{Q}_{\geq 0}\times\mathbb{Z}_{\geq 0}\smallsetminus\{(0,0)\}$.

The skew polynomial ring $K(x)[d_q]$ is euclidean with respect to $\deg_{d_q}$. It follows that, if we have a formal power series y solution of a q-difference operator, we can find a q-difference operator $\mathcal{L}$ killing y and such that $\deg_{d_q}\mathcal{L}$ is minimal. All the other linear q-difference operators killing y, minimal with respect to $\deg_{d_q}$, are of the form $f(x)\mathcal{L}$, with $f(x)\in K(x)$. By abuse of language, we will call the minimal degree operator $\mathcal{L}\in K[x,d_q]$ (resp. $K[x,\sigma_q]$) with no common factors in the coefficients *the* minimal operator killing y.

Remark 12.1. Let $y(x)\in K[[x]]$ be a formal power series solution of the linear q-difference operator $\mathcal{L}_q=\sum_{i=0}^{\nu}a_i(x)\sigma_q^i$. We choose $\mathcal{L}_q$ such that $\deg_{\sigma_q}\mathcal{L}_q$ is minimal. Then for all positive integers r the operator $\mathcal{L}_{q^{1/r}}=\sum_{i=0}^{\nu}a_i(x)\sigma_{q^{1/r}}^{ir}$ is the minimal $q^{1/r}$-difference operator killing $y(x)$. Moreover if λ is a slope of $NRP_{\sigma_q}(\mathcal{L}_q)$ (resp. $NRP_{d_q}(\mathcal{L}_q)$) then λ/r is a slope of $NRP_{\sigma_q}\left(\mathcal{L}_{q^{1/r}}\right)$ (resp. $NRP_{d_q}\left(\mathcal{L}_{q^{1/r}}\right)$). In fact, let $\mathcal{L}$ be a $q^{1/r}$-difference operator killing $y(x)$, minimal with respect to $\deg_{\sigma_{q^{1/r}}}$. Then $\mathcal{L}_{q^{1/r}}$ is a factor of $\mathcal{L}$ in $K(x)[\sigma_{q^{1/r}}]$, hence $\deg_{\sigma_{q^{1/r}}}\mathcal{L}\leq r\deg_{\sigma_q}\mathcal{L}_q$. On the other side we have:

$$\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_{q^{1/r}}^i(y)\geq\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_{q^{1/r}}^{ir}(y)=\dim_{K(x)}\sum_{i\geq 0}K(x)\sigma_q^i(y),$$

therefore $\deg_{\sigma_{q^{1/r}}}\mathcal{L}\geq r\deg_{\sigma_q}\mathcal{L}_q$.

We recall the statement of Corollary 4.4, which is the starting point for this second part of the paper:

Proposition 12.2. *Let $F\in K[[x]]$ be a global q-Gevrey series of orders $(0,0)$ and $\mathcal{L}\in K[x,d_q]$ the minimal q-difference operator such that $\mathcal{L}F=0$. Then $\mathcal{L}$ is regular singular.*

Using the formal q-Fourier transformations introduced in the previous section, we will deduce the structure theorems below from Proposition 12.2.

Theorem 12.3. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(-s_1,-s_2)$, with $(s_1,s_2)\in\mathcal{Z}$ and $\mathcal{L}\in K\left[x,d_q\right]$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$. Then $\mathcal{L}$ has the following properties:*
- the set of finite slopes of the Newton Polygon $NRP_{d_q}(\mathcal{L})$ is $\{-1/(s_1+s_2),0\}$;
- for all $\xi\in K^*$, the q-difference operator $\mathcal{L}$ has a basis of solutions in $K[[x-\xi]]_q$.

[5]or equivalently, that $NRP_{d_q}(S\circ\mathcal{L})$ does not have any positive slope.

Proof. Let us write the formal power series F in the form:

$$F=\sum_{n=0}^{\infty}\frac{a_n}{\left(q^{\frac{n(n-1)}{2}}\right)^{s_1}\left([n]_q^!\right)^{s_2}}x^n\,,$$

where $\sum_{n=0}^{\infty}a_nx^n$ is a G_q-function. Let $\widetilde{F}(x)=\sum_{n=0}^{\infty}a_nx^{n+s_2}$; then the series $\widetilde{F}$ has finite size, therefore there exists a regular singular q-difference operator $\mathcal{L}\in K[x,\sigma_q]$ such that $\mathcal{L}\widetilde{F}=0$. The polygon $NRP_{\sigma_q}(\mathcal{L})$ has only the zero slope (apart from the infinite slopes).

Let $\mathcal{S}$ be the symmetry with respect to the origin:

$$\mathcal{S}:\quad \begin{array}{ccc} x & \longmapsto & 1/z \\ \sigma_q & \longmapsto & \sigma_p \end{array}.$$

Remark that the operator $\mathcal{F}_{q^+}^{-1}\circ\mathcal{S}\,(\mathcal{L})$ kill the formal power series $\sum_{n=0}^{\infty}\frac{a_n}{[n]_q^!}x^{n+s_2-1}$. The polygon $NRP_{d_p}(\mathcal{S}(\mathcal{L}))$ is obtained by $NRP_{d_q}(\mathcal{L})$ applying a symmetry with respect to the line $v=0$. It follows from Proposition 10.4 that the set of finite slopes of $NRP_{d_q}\left(\mathcal{F}_{q^+}^{-1}\circ\mathcal{S}\,(\mathcal{L})\right)$ is $\{0,-1\}$. Iterating s_2 times this reasoning, we obtain a q-difference operator $\widetilde{\mathcal{L}}=\mathcal{F}_{q^+}^{-1}\circ\mathcal{S}\circ\cdots\circ\mathcal{F}_{q^+}^{-1}\circ\mathcal{S}\,(\mathcal{L})$, such that the set of finite slopes of $NRP_{d_q}\left(\widetilde{\mathcal{L}}\right)$ is $\{0,-1/s_2\}$. We obtain:

$$\widetilde{\mathcal{L}}\left(\sum_{n=0}^{\infty}\frac{a_n}{\left([n]_q^!\right)^{s_2}}x^n\right)=0\,.$$

Because of §8.2, we can now suppose that s_1 is actually a positive integer. We conclude the proof applying the same argument to $\overline{\mathcal{L}}=\left(\mathcal{F}_{q^\#}^{-1}\circ\circ\mathcal{S}\right)\circ\cdots\circ\left(\mathcal{F}_{q^\#}^{-1}\circ\mathcal{S}\right)\left(\sigma_q^n\circ\widetilde{\mathcal{L}}\circ x^{s_1}\right)$, for a suitable $n\in\mathbb{Z}_{\geq 0}$, and to the Newton-Ramis Polygon defined with respect to σ_q. We know that $\overline{\mathcal{L}}F=0$.

The operator $\mathcal{L}$ is a factor of $\overline{\mathcal{L}}$ in $K(x)[\sigma_q]$. We know (*cf.* for instance [Sau04]) that the slopes of the Newton Polygon of $\mathcal{L}$ at zero (resp. ∞) are slopes of the Newton Polygon of $\overline{\mathcal{L}}$ at zero (resp. ∞). To obtain the desired result on the slopes of $NRP_{d_q}(\mathcal{L})$ one has to notice that $\overline{\mathcal{L}}$ must have a positive slope at ∞ because of [Ram92, Theorem 4.8]. As far as $\xi\in K^*$ is concerned, the operator $\overline{\mathcal{L}}$ has a basis of solutions at ξ in $K[[x-\xi]]_q$ (*cf.* Propositions 11.1 and 11.3), therefore the same is true for $\mathcal{L}$. □

Proposition 10.3 implies that for a global q-Gevrey series of orders $(-s_1,0)$ we have actually proved a more precise result:

Theorem 12.4. *Under the hypothesis of the previous theorem, we assume that $s_2=0$. Then $\mathcal{L}$ has the following properties:*
- the set of finite slopes of $NP_{\sigma_q}(\mathcal{L})$ is $\{0,-1/s_1\}$;
- for all $\xi\in K^*$, the q-difference operator $\mathcal{L}$ has a basis of solutions in $K[[x-\xi]]_q$.

Changing q in q^{-1} we get the corollary:

Corollary 12.5. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(s_1,-s_2)$, with $(s_1,s_2)\in\mathbb{Q}\times\mathbb{Z}$, such that $s_1\geq s_2\geq 0$ and either $s_1\neq s_2$ or $s_2\neq 0$. Let $\mathcal{L}\in K[x,\sigma_q]$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$. Then $\mathcal{L}$ has the following properties:*
- the set of finite slope of $NP_{d_p}(\mathcal{L})$ is $\{0,1/s_1\}$
- for all $\xi\in K^*$, the q-difference operator $\mathcal{L}$ has a basis of solutions in $K[[x-\xi]]_p$.

Proof. It follows by Proposition 8.4, taking into account that when one changes q in q^{-1}, the slopes of the Newton Polygon change sign. □

Following [And00b] we can characterize the apparent singularities of such a q-difference equation:

Theorem 12.6. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(-s_1,-s_2)$, with $(s_1,s_2)\in\mathcal{Z}$. We fix a point $\xi\in K^*$. For all $v\in\mathcal{P}$ such that $|q|_v>1$ we suppose that the v-adic function $F(x)$ has a zero at ξ. Let $\mathcal{L}\in K[x,d_q]$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$. Then $\mathcal{L}$ has a basis of solution in*

$$(x-\xi)K[[x-q\xi]]_q=\left\{\sum_{n=1}^{\infty}a_n(x-\xi)_n:a_n\in K\right\}.$$

The proof is based on the following lemma, which is an analogue of [And00b, Lemme 2.1.2] (*cf.* also [And00b, Lemma 4.4.2]).

Lemma 12.7. *Let F be a global q-Gevrey series of orders $(-s_1,-s_2)$, with $s_1,s_2\in\mathbb{Q}_{\geq 0}\times\mathbb{Z}_{\geq 0}$. We fix a point $\xi\in K^*$. For all $v\in\mathcal{P}$ such that $|q|_v>1$ we suppose that the v-adic entire function $F(x)$ has a zero at ξ. Then $G=(x-\xi)^{-1}F$ is a global q-Gevrey series of orders $(-s_1,-s_2)$.*

Proof of Theorem 12.6. We fix some notation:

$$F=\sum_{n=0}^{\infty}\frac{a_n}{q^{s_1\frac{n(n-1)}{2}}[n]_q^{!\,s_2}}x^n\,,\ G=\sum_{n=0}^{\infty}\frac{b_n}{q^{s_1\frac{n(n-1)}{2}}[n]_q^{!\,s_2}}x^n\,,$$

$$\widetilde{h}(n,v,F)=\sup_{s\leq n}|a_s|_v \text{ and } \widetilde{h}(n,v,G)=\sup_{s\leq n}|b_s|_v\,.$$

Since $\frac{1}{x-\xi}=-\sum_{n\geq 0}\frac{x^n}{\xi^{n+1}}$, we obtain:

$$b_n=-\sum_{k=0}^{n}\left(q^{\frac{n(n-1)}{2}-\frac{k(k-1)}{2}}\right)^{s_1}\left(\frac{[n]_q^!}{[k]_q^!}\right)^{s_2}\xi^{k-n-1}a_k$$

and therefore:

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{|q|_v\leq 1}\widetilde{h}(n,v,G)\leq\limsup_{n\to\infty}\frac{1}{n}\sum_{|q|_v\leq 1}\widetilde{h}(n,v,F)+\sum_{|q|_v\leq 1}|\xi|_v\,.$$

To conclude it is enough to prove that G is a local q-Gevrey series of order s_1+s_2 for all $v\in\mathcal{P}$ such that $|q|_v>1$. This follows from [Ram92, Prop. 2.1], since F and G have the same growth at ∞, because F has a zero at ξ. □

Proof. Let $G=(x-\xi)^{-1}F$ and $\mathcal{L}$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$; then $\mathcal{L}\circ(x-\xi)$ is the minimal linear q-difference operator such that $\mathcal{L}\circ(x-\xi)(G)=0$. By Lemma 12.7 and Theorem 12.3, $\mathcal{L}\circ(x-\xi)$ has a basis of solution in $K[[x-q\xi]]_q$, therefore the operator $\mathcal{L}$ has a basis of solution in $(x-\xi)K[[x-q\xi]]_q$. □

Once again, switching q into q^{-1} we obtain the corollary:

Corollary 12.8. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(s_1,-s_2)$, with $s_1,s_2\in\mathbb{Q}\times\mathbb{Z}$, $s_1\geq s_2\geq 0$ and either $s_1\neq s_2$ or $s_2\neq 0$. We fix a point $\xi\in K^*$. For all $v\in\mathcal{P}$ such that $|q|_v<1$ we suppose that the v-adic function $F(x)$ has a zero at ξ. Let $\mathcal{L}\in K[x,d_q]$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$. Then $\mathcal{L}$ has a basis of solution in*

$$(x-\xi)K[[x-p\xi]]_p\,.$$

Proof. It follows from Proposition 8.4 and Theorem 12.6. □

We conclude the section with an example:

Example 12.9. Let us consider the q-exponential series $E_q(x)=\sum_{n\geq 0}\frac{x^n}{[n]_q^!}$, solution of the equation $d_qy=y$. A classical formula (*cf.* [GR90, 1.3.16]) says that for $|q|_v>1$ the series $E_q(x)$ can be written as an infinite product:

$$E_q(x)=\left(-x(1-q^{-1});q^{-1}\right)_\infty:=\prod_{k=0}^{\infty}\left(1-x\frac{1-q}{q^{k+1}}\right)\,,$$

hence $E_q(\frac{q}{1-q})=0$ for all v such that $|q|_v>1$. Let us consider formal q-series:

$$G(x)=\frac{E_q(x)}{x-\frac{q}{1-q}}=\frac{q-1}{q}E_q\left(\frac{x}{q}\right)\,.$$

Obviously, $qd_qG(x)-G(x)=0$ and actually:

$$(d_q-1)\circ\left(x-\frac{q}{1-q}\right)G(x)=\left(x-\frac{1}{1-q}\right)(qd_q-1)\,G(x)=0\,.$$

Since $\sum_{n\geq 0}\frac{q^{-n}}{[n]_q^!}T_n^q\left(x,\frac{q^2}{1-q}\right)\in K[[x-\frac{q^2}{1-q}]]_q$ is a formal solution of $qd_qy=y$, the series

$$\left(x-\frac{q}{1-q}\right)\sum_{n\geq 0}\frac{q^{-n}}{[n]_q^!}T_n^q\left(x,\frac{q^2}{1-q}\right)\in\left(x-\frac{q}{1-q}\right)K\left[\left[x-\frac{q^2}{1-q}\right]\right]_q$$

is a formal solution of $d_qy=y$.

13. An irrationality result for global q-Gevrey series of negative orders

In this section we are going to give a simple criteria to determine the q-orbits where a global q-Gevrey series does *not* satisfy the hypothesis of Theorem 12.6. We will deduce an irrationality result for values of a global q-Gevrey series $F(x)\in K[[x]]\smallsetminus K[x]$ of negative orders.

Remark 13.1. The arithmetic Gevrey series theory in the differential case has applications to transcendence theory (*cf.* [And00b]). In the global q-Gevrey series framework this can not be true, since the set of global q-Gevrey series has only a structure of $\overline{k}(q)$-vector space. We mean that the product of two global q-Gevrey series of nonzero orders doesn't need to be a global q-Gevrey series, as the following example shows:

$$e_q(x)^2=\left(\sum_{n=0}^{\infty}\frac{x^n}{[n]_q^!}\right)^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q\right)\frac{x^n}{[n]_q^!}\,.$$

In fact, because of the estimate at the cyclotomic places $e_q(x)^2$ should be a global q-Gevrey series of order $(0,-1)$, while the local q-Gevrey order at places $v\in\mathcal{P}_\infty$ such that $|q|_v>1$ is 2. For this reason a global q-Gevrey series theory can only have applications to the irrationality theory.

Let

$$\mathcal{L}=a_\nu(x)\sigma_q^\nu+\cdots+a_1(x)\sigma_q+a_0(x)\in K\left[x,\sigma_q\right]\,,$$

and let $u_0,\dots,u_{\nu-1}$ a basis of solution of $\mathcal{L}$ is a convenient q-difference algebra extending $K(x)$. The *Casorati matrix*

$$\mathcal{U}=\begin{pmatrix}u_0&\cdots&u_{\nu-1}\\ \sigma_qu_0&\cdots&\sigma_qu_{\nu-1}\\ \vdots&\ddots&\vdots\\ \sigma_q^{\nu-1}u_0&\cdots&\sigma_q^{\nu-1}u_{\nu-1}\end{pmatrix},$$

is a fundamental solution of the q-difference system

$$\sigma_q\mathcal{U}=\left(\begin{array}{c|ccc}0&&&\\ \vdots&&\mathbb{I}_{\nu-1}&\\ 0&&&\\ \hline -\frac{a_0(x)}{a_\nu(x)}&-\frac{a_1(x)}{a_\nu(x)}&\cdots&-\frac{a_{\nu-1}(x)}{a_\nu(x)}\end{array}\right)\mathcal{U}\,,$$

so that $\mathcal{C}=\det\mathcal{U}$ is solution of the equation:

$$\sigma_q\mathcal{C}=(-1)^\nu\frac{a_0(x)}{a_\nu(x)}\mathcal{C}\,.$$

Notice that the "q-Wronskian lemma" (*cf.* for instance [DV02, §1.2]) implies that the determinant of the Casorati matrix of a basis of solutions of an operator $\mathcal{L}$ is nonzero.

Proposition 13.2. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(-s_1,-s_2)$, with $s_1,s_2\in\mathcal{Z}$. We fix a point $\xi\in K^*$. Let $\mathcal{L}=a_\nu(x)\sigma_q^\nu+\cdots+a_1(x)\sigma_q+a_0(x)\in K\left[x,\sigma_q\right]$ be the minimal q-difference operator such that $\mathcal{L}F=0$. If $F(x)$ has a zero at ξ for all v such that $|q|_v>1$, then there exists an integer $m\geq 0$ such that $q^m\xi$ is a zero of $a_0(x)$.*

Proof. The determinant of the Casorati matrix of a basis of solutions of $\mathcal{L}$ satisfies the equation

$$y(qx)=(-1)^\nu\frac{a_\nu(x)}{a_0(x)}y(x)\,.$$

On the other hand we know that $\mathcal{L}$ has a basis of solution $u_0,\dots,u_{\nu-1}\in(x-\xi)K[[x-q\xi]]$. This means that the u_i's are formal series of the form $\sum_{n\geq 1}a_nT_n^q(x,\xi)$, for some $a_n\in K$. Since $(qx-\xi)=q(x-q^{n-1}\xi)+(q^n-1)\xi$, one obtain that

$$\sigma_q\left(\sum_{n\geq 1}a_nT_n^q(x,\xi)\right)=qa_1+\sum_{n\geq 1}\left(q^na_n+q^{n+1}a_{n+1}\xi(q^n-1)\right)\right)T_n^q(x,\xi)\,.$$

This implies that the determinant $\mathcal{C}$ of the Casorati matrix of $u_0,\dots,u_{\nu-1}$ is an element of $(x-\xi)K[[x-q\xi]]_q$. Let $m\geq 1$ be the larger integer such that $\mathcal{C}\in T_m^q(x,\xi)K[[x-q^m\xi]]_q$. The formula above implies that $\sigma_q\mathcal{U}\in T_{m-1}^q(x,\xi)K[[x-q^{m-1}\xi]]_q\smallsetminus T_m^q(x,\xi)K[[x-q^m\xi]]_q$, and therefore that $q^{m-1}\xi$ is a zero of $a_0(x)$. □

In the same way we can prove the following result:

Corollary 13.3. *Let $F\in K[[x]]\smallsetminus K[x]$ be a global q-Gevrey series of orders $(s_1,-s_2)$, with $s_1,s_2\in\mathbb{Q}\times\mathbb{Z}$, $s_1\geq s_2\geq 0$ and either $s_1\neq s_2$ or $s_2\neq 0$. We fix a point $\xi\in K^*$. Let $\mathcal{L}=a_\nu(x)\sigma_q^\nu+\dots+a_1(x)\sigma_q+a_0(x)\in K[x,\sigma_q]$ be the minimal linear q-difference operator such that $\mathcal{L}F=0$. If $F(x)$ has a zero at ξ for all $v\in\mathcal{P}$ such that $|q|_v<1$ then there exists an integer $m\leq-\nu$ such that $q^m\xi$ is a zero of $a_\nu(x)$.*

Proof. It follows from Proposition 8.4 that $F(x)$ is a global q^{-1}-Gevrey series of negative orders $(-(s_1-s_2),-s_2)$ and the minimal linear q^{-1}-difference operator killing $F(x)$ is $a_\nu(q^{-\nu}x)+\dots+a_1(q^{-\nu}x)\sigma_{q^{-1}}^{\nu-1}+a_0(q^{-\nu}x)\sigma_{q^{-1}}^{\nu}$. □

Example 13.4. Let us consider the field $K=k(q)$ and the Tchakaloff series:

$$T_q(x)=\sum_{n\geq 0}\frac{x^n}{q^{n(n-1)/2}}\,.$$

Together with $E_q(x)$, $T_q(x)$ is a q-analogue of the exponential function. The minimal linear q-difference equation killing $T_q(x)$ is

$$\mathcal{L}=(\sigma_q-1)\circ(\sigma_q-qx)=(\sigma_q-q^2x)\circ(\sigma_q-1)=\sigma_q^2-(1+q^2x)\sigma_q+q^2x\,.$$

Notice that $1,T_q(x)$ is a basis of solutions of $\mathcal{L}$ at zero. We conclude that $T_q(\xi)\neq 0$ for all $\xi\in K^*$, as the value a q^{-1}-adic entire analytic function, *i.e.* the hypothesis of Theorem 12.6 are never satisfied.

In particular, let $K=k(\widetilde{q})$, where $\widetilde{q}^r=q$ for some positive integer r. For any $\xi\in k(\widetilde{q})$, $\xi\neq 0$, the $\widetilde{q}^{-1}$-adic value $T_q(\xi)$ of $T_q(x)$ at ξ can be formally written as a Laurent series in $k((\widetilde{q}^{-1}))$, which is the completion of $k(\widetilde{q})$ at the $\widetilde{q}^{-1}$-adic place. The theorem above says that $T_q(\xi)$ cannot be the expansion of a rational function in $k(\widetilde{q})$. In fact, if it was, there would exists $c\in k(\widetilde{q})$ such that $T_q(x)+c$ has a zero at ξ and is solution of $\mathcal{L}$. This would imply that $\mathcal{L}$ has a basis of solutions having a zero at ξ, against the fact that the constants are solution of $\mathcal{L}$.

As in [And00b], we can also deduce a Lindemann-Weierstrass type statement:

Corollary 13.5. *Let $K=k(\widetilde{q})$, where $\widetilde{q}$ is a root of q. We consider the q-exponential function $e_q(x)=\sum_{n\geq 0}\frac{x^n}{[n]_q^!}$ and a set of element $a_1,\dots,a_r\in K$, which are multiplicatively independent modulo $q^{\mathbb{Z}}$ (*i.e.* $\alpha_1^{\mathbb{Z}}\cdots\alpha_r^{\mathbb{Z}}\cap q^{\mathbb{Z}}=\{1\}$). Then the Laurent series $e_q(a_1\xi),\dots,e_q(a_r\xi)\in k((\widetilde{q}^{-1}))$ are linearly independent over $k(\widetilde{q})$ for any $\xi\in K^*$.*

Proof. It is enough to notice that $e_q(a_1x),\dots,e_q(a_rx)$ is a basis of solutions of the operator

$$(d_q-a_1)\circ\dots\circ(d_q-a_r)\,.$$

If there exist $\lambda_1,\dots,\lambda_r\in K$ such that $\lambda_1e_q(\alpha_1\xi)+\dots+\lambda_re_q(\alpha_r\xi)=0$, then $e_q(\alpha_i\xi)=0$ for any $i=1,\dots,r$, because of Theorem 12.6. Since $e_q(x)$ satisfies the equation $y(qx)=(1+(q-1)x)e_q(x)$, we deduce that $\xi\in\frac{q^{\mathbb{Z}\geq 1}}{(1-q)\alpha_i}$, for any $i=1,\dots,r$. The last assertion would imply that $\alpha_i\alpha_j^{-1}\in q^{\mathbb{Z}}$ for any pair of distinct i,j, against the assumption. □

We can deduce by Theorem 12.6 an irrationality result for all global q-Gevrey series $F(x)$ such that zero is not a slope of the Newton Polygon at ∞ of the minimal q-difference operator that kills $F(x)$:

Theorem 13.6. *Let $\overline{k(q)}$ be a fixed algebraic closure of $k(q)$ and $\widetilde{K} \subset \overline{k(q)}$ the maximal extension of $k(q)$ such that the q^{-1}-adic norm of $k(q)$ extends uniquely to $\widetilde{K}$.*

Let $F(x) \in \widetilde{K}[[x]] \smallsetminus \widetilde{K}[x]$ be a global q-Gevrey series of orders $(-s_1, -s_2)$, with $(s_1, s_2) \in \mathcal{Z}$, and $\mathcal{L}$ the minimal linear q-difference operator such that $\mathcal{L}F(x) = 0$. We suppose that zero is not a slope of $\mathcal{L}$ at ∞. Then for all $\xi \in K^*$ the value $F(\xi)$ of the q^{-1}-adic analytic entire function $F(x)$ is not an element of $\widetilde{K}$ (but of its $\widetilde{q}^{-1}$-adic completion).

Before proving the theorem, we give an example, which illustrates the proof:

Example 13.7. Let us consider the q-analogue of a Bessel series

$$B_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q^{!\,2}}\,.$$

The series $B_q(x)$ is solution of the linear q-difference operator $(xd_q)^2 - x$ that can be written also in the form:

$$\mathcal{L} = \sigma_q^2 - 2\sigma_q + (1 - (q-1)^2 x)\,.$$

There is a unique factorization of a linear q-difference operator linked to the slopes of its Newton Polygon (*cf.* [Sau04]): we deduce that $\mathcal{L}$ is the minimal q-difference operator killing $B_q(x)$ from the fact that the only slope of the Newton-Polygon of $\mathcal{L}$ at ∞ is $-1/2$. We conclude that $B_q(\xi) = 0$ for all v such that $|q|_v > 1$, with $\xi \in \mathbb{P}^1(K)$, implies $\xi = q^m/(q-1)^2$ for some integer $m \geq 2$.

Let $K = k(\widetilde{q})$, with $\widetilde{q}^r = q$ for some positive integer r. In this case the $\widetilde{q}^{-1}$-adic norm is the only one such that $|q|_v > 1$. For any $c \in K$ we have:

$$(q\sigma_q - 1) \circ \mathcal{L}(B_q(x) + c) = 0\,.$$

One notices that the slopes of the Newton Polygon of $(q\sigma_q - 1) \circ \mathcal{L}$ at ∞ are $\{0, -1/2\}$, therefore we deduce from the uniqueness of the factorization that $(q\sigma_q - 1) \circ \mathcal{L}$ is the minimal q-difference operator killing $B_q(x) + c$. Since constants are solutions of $(q\sigma_q - 1) \circ \mathcal{L}$, Theorem 12.6 implies that no solution of $(q\sigma_q - 1) \circ \mathcal{L}$ can have a zero at any point $\xi \in K^*$ as $\widetilde{q}^{-1}$-adic holomorphic functions. This means that the function $B_q(x) + c$ cannot have a zero as a $\widetilde{q}^{-1}$-adic analytic function at $\xi \in K^*$, which means that $B_q(x)$ takes values in $k((\widetilde{q}^{-1})) \smallsetminus k(\widetilde{q})$ at each $\xi \in K^*$.

Proof of Theorem 13.6. Let $c \in \widetilde{K}$, $c \neq 0$, $G(x) = F(x) + c$, $\mathcal{L} = \sum_{i=1}^{\nu} a_i(x) d_q^i \in \widetilde{K}[x, d_q]$ (resp. $\mathcal{N} = \sum_{j=1}^{\mu} b_j(x) d_q^j \in \widetilde{K}[x, d_q]$) be the minimal q-difference operator killing $F(x)$ (resp. $G(x)$). Of course we may assume that $a_i(x), b_j(x) \in \widetilde{K}(x)$ and $a_\nu(x) = b_\mu(x) = 1$, and that everything is defined over a finite extension $K \subset \widetilde{K}$ of $k(q)$.

Since:

$$\left(d_q - \frac{d_q(a_0)(x)}{a_0(x)}\right) \circ \mathcal{L}(G(x)) = 0 \text{ and } \left(d_q - \frac{d_q(b_0)(x)}{b_0(x)}\right) \circ \mathcal{N}(F(x)) = 0\,,$$

we must have $\nu - 1 \leq \mu \leq \nu + 1$. Let us suppose first $\nu = \mu$. Then

$$\left(d_q - \frac{d_q(a_0)(x)}{a_0(x)}\right) \circ \mathcal{L} = \left(d_q - \frac{d_q(b_0)(x)}{b_0(x)}\right) \circ \mathcal{N}$$

since they have the same set of solutions and they are both monic operators. By hypothesis, zero is not a slope of the Newton Polygon of $\mathcal{L}$ at ∞, while $\left(d_q - \frac{d_q(a_0)(x)}{a_0(x)}\right)$ has only the zero slope at ∞: we conclude by the uniqueness of the factorization that $\mathcal{L} = \mathcal{N}$. We remark that the equality $\mathcal{L} = \mathcal{N}$ implies that constants are solutions of $\mathcal{L}$ and that $\mathcal{L}$ has a zero slope at ∞, hence we obtain a contradiction. So either $\mu = \nu - 1$ or $\mu = \nu + 1$. If $\mu = \nu - 1$, then

$$\mathcal{L} = \left(d_q - \frac{d_q(b_0)(x)}{b_0(x)}\right) \circ \mathcal{N}$$

since both $\mathcal{L}$ and $\mathcal{N}$ are monic. Once again, constants are solution of $\mathcal{L}$ and this is a contradiction. Finally, we have necessarily $\mu = \nu + 1$ and

$$\mathcal{N} = \left(d_q - \frac{d_q(b_0)(x)}{b_0(x)}\right) \circ \mathcal{L}\,.$$

Let us suppose that there exists $\xi \in K^*$, such that $F(x)$ takes a value in K at ξ, as $\widetilde{q}^{-1}$-adic analytic function. Then all the solutions of $\mathcal{N}$ would have a zero at ξ against the fact that the constants are solutions of $\mathcal{N}$, hence $F(\xi) \neq 0$ is not in K. □

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