Alternative proof of [DV02, Th. 6.2.2, 1)]

Lucia Di Vizio *

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Introduction

We recall the precise statement. Let $\mathcal{M} = (M, \Phi_q)$ be a q-difference module of finite rank μ over a q-difference algebra $\mathcal{A} \subset K(x)$, essentially of finite type over the ring of integers $\mathcal{V}_K$ of the number field K[1]. By definition, $\mathcal{M}$ is regular singular at zero if $\mathcal{M} \otimes_{\mathcal{A}} K((x))$ has a basis over $K((x))$ in which Φ_q acts through a matrix in $Gl_\mu(K)$. We say that $\mathcal{M}$ is regular singular if it is regular singular at 0 and at ∞, *i.e.* if it is regular singular at 0 after a variable change $z = 1/x$.

Theorem 1 *Let the set Σ_{nilp} of finite places v of K such that $\mathcal{M}$ has unipotent reduction[2] modulo ϖ_v be infinite. Then the q-difference module $\mathcal{M}$ is regular singular.*

In [DV02, Th. 6.2.2] we used a result of Pragmaan [Pra83] (*cf.* also [DV02, 1.4.4]) about the formal classification of singularities of finite difference operators. Here we follow [Kat70, §11].

Proof of theorem 1

Of course it is enough to prove that 0 is a regular singular point for $\mathcal{M}$.

Proposition 2 *[Sau00, Annexe B] Let $\mathcal{M}_{K(x)} = \mathcal{M} \otimes_{\mathcal{A}} K(x)$. The following fact are equivalent:*
1) $\mathcal{M}$ is regular singular at 0.
2) The action of Φ_q on one (and actually on any any) cyclic basis[3] $\underline{e}$ of $\mathcal{M}_{K(x)}$

$$\Phi_q \underline{e} = \underline{e} \left(\begin{array}{ccc|c} 0 & \cdots & 0 & a_0(x) \\ \hline 1 & & 0 & a_1(x) \\ & \ddots & & \vdots \\ 0 & & 1 & a_{\mu-1}(x) \end{array} \right) \tag{1}$$

is such that $a_0(x), \ldots, a_{\mu-1}(x) \in K(x)$ have no poles at 0 and $a_0(0) \neq 0$.

Let $d \in \mathbb{N}$ be equal to 1 or to a multiple of $\mu!$ and let L be a finite extension of K containing an element $\widetilde{q}$ such that $\widetilde{q}^d = q$. We consider the field extension $K(x) \hookrightarrow L(t)$, $x \mapsto t^d$: the field $L(t)$ has a natural structure of $\widetilde{q}$-difference algebra extending the q-difference structure of $K(x)$. It follows by the previous proposition that:

Corollary 3 *The q-difference module $\mathcal{M}_{K(x)}$ is regular singular at $x = 0$ if and only if the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$ is regular singular at $t = 0$.*

Proof. It is enough to notice that if $\underline{e}$ is a cyclic basis for $\mathcal{M}_{K(x)}$, then $\underline{e} \otimes 1$ is a cyclic basis for $\mathcal{M}_{L(t)} \cong \mathcal{M}_{K(x)} \otimes_{K(x)} L(t)$ and $\Phi_{\widetilde{q}}(\underline{e} \otimes 1) = \Phi_q(\underline{e}) \otimes 1$. ■

In the next lemma we construct a rational gauge transformation that allows to avoid the use of the much stronger Praagmann [Pra83] result:

*École Normale Supérieure, 45 rue d'Ulm, 75005 Paris, France. e-mail: lucia.di.vizio@ens.fr

[1] for notation *cf.* [DV02, §1] and in particular [DV02, 1.1.2], [DV02, 1.1.5]

[2] for the definition of unipotent reduction *cf.* [DV02, §5]

[3] *cf.* [DV02, §1.3]

Lemma 4 *There exists a basis $\underline{f}$ of the $\widetilde{q}$-difference module $\mathcal{M}_{L(t)}$, such that $\Phi_{\widetilde{q}}\underline{f} = \underline{f}B(t)$, with $B(t) \in Gl_\mu(L(t))$, and an integer k such that*

$$\begin{cases} B(t) = \dfrac{B_k}{t^k} + \dfrac{B_{k-1}}{t^{k-1}} + \dots, \text{ as an element of } Gl_\mu(L((t))); \\ B_k \text{ is a constant non nilpotent matrix.} \end{cases} \tag{2}$$

Proof. We follow [Kat70, §11]. If 0 is a regular singular point for $\mathcal{M}_{K(x)}$, it follows from the previous proposition that it is enough to chose $d = 1$ and $K(x) = L(t)$. Therefore let us suppose that $\mathcal{M}_{K(x)}$ is not regular singular at 0 and fix a cyclic basis $\underline{e}$ of $\mathcal{M}_{K(x)}$: then $\Phi_q\underline{e} = \underline{e}A(x)$, with $A(x)$ of the form (1). Let

$$k = \max_{j=0,\dots,\mu-1} \left(-\frac{1}{\mu - j} \mathrm{ord}_{t=0} a_j(t^d) \right) \neq 0\,. \tag{3}$$

Notice that the rational number k is actually an integer. Consider the basis $\underline{f} = \underline{e}F(t)$ of $\mathcal{M}_{L(t)}$ with $F(t) = \mathrm{diag}(1, t^k, \dots, t^{(m-1)k})$. Then

$$\Phi_{\widetilde{q}}\underline{f} = \underline{f}\left[F(t)^{-1}A(x)F(\widetilde{q}t)\right] = \underline{f} \left(\begin{array}{ccc|c} 0 & \cdots & 0 & a_0(t^d)\widetilde{q}^{(\mu-1)k}t^{(\mu-1)k} \\ \hline t^{-k} & & 0 & a_1(t^d)\widetilde{q}^{(\mu-1)k}t^{(\mu-2)k} \\ & \ddots & & \vdots \\ 0 & & t^{-k} & a_{\mu-1}(t^d)\widetilde{q}^{(\mu-1)k} \end{array} \right).$$

It follows from (3) that $\mathrm{ord}_{t=0}a_j(t^d)\widetilde{q}^{(\mu-1)k}t^{(\mu-j-1)k} \geq -k$ and that we have an equality for at least one $j = 0, \dots, \mu - 1$. hence that

$$\Phi_{\widetilde{q}}\underline{f} = \underline{f}\left(\frac{B_k}{t^k} + h.o.t\right), \text{ with } B_k = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & b_0 \\ \hline 1 & & 0 & b_1 \\ & \ddots & & \vdots \\ 0 & & 1 & b_{\mu-1} \end{array} \right)$$

and $b_0, \dots, b_{m-1} \in L$ not all equal to 0. One can verify recursively that $\det(t - B_k) = t^\mu - \sum_{i=0,\dots,\mu-1} b_i t^i$ and hence that B_k is not nilpotent. ■

Let $\mathcal{B} \subset L(t)$ be any $\widetilde{q}$-difference algebra essentially of finite type over the ring of integers $\mathcal{V}_L$ of L, containing the entries of $B(x)$. Then there exists a $\mathcal{B}$-lattice $\mathcal{N}$ of $\mathcal{M}_{L(t)}$ inheriting the $\widetilde{q}$-difference module structure from $\mathcal{M}_{L(t)}$ and having the following properties:

1. $\mathcal{N}$ has unipotent reduction modulo infinitely many finite place of L, namely almost all the places dividing a place in Σ_{nilp};
2. there exists a basis $\underline{f}$ of $\mathcal{N}$ over $\mathcal{B}$ such that $\Phi_{\widetilde{q}}\underline{f} = \underline{f}B(t)$ and $B(t)$ verifies (2).

Iterating the operator $\Phi_{\widetilde{q}}$ we obtain:

$$\Phi_{\widetilde{q}}^m(\underline{f}) = \underline{f}B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{m-1}t) = \underline{f}\left(\frac{B_k^m}{q^{\frac{km(m-1)}{2}}x^{mk}} + h.o.t.\right)$$

We know that for almost any finite place w of L for whom we have unipotent reduction the matrix $B(t)$ verifies

$$\left(B(t)B(\widetilde{q}t)\cdots B(\widetilde{q}^{\kappa_w - 1}t) - 1\right)^{n(w)} \equiv 0 \bmod \varpi_w\,, \tag{4}$$

where ϖ_w is an uniformizer of the palce w, κ_w is the order $\widetilde{q}$ modulo ϖ_w and $n(w)$ is a convenient positive integer. Suppose that $k \neq 0$. Then $B_k^{\kappa_w} \equiv 0$ modulo ϖ_w, for infinitely many w, and hence that B_k is a nilpotent matrix, in contradiction with lemma 4. So necessarily $k = 0$.

Finally we have $\Phi_{\widetilde{q}}(\underline{f}) = \underline{f}(B_0 + h.o.t)$. It follows from (4) that B_0 is actually invertible, which implies that $\mathcal{M}_{L(t)}$ is regular singular at 0. Corollary 3 allows to conclude.

References

[DV02] Lucia Di Vizio. Arithmetic theory of q-difference equations. The q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Inventiones Mathematicae*, 150(3):517–578, 2002.

[Kat70] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.

[Pra83] C. Praagman. The formal classification of linear difference operators. *Nederl. Akad. Wetensch. Indag. Math.*, 45(2):249–261, 1983.

[Sau00] Jacques Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Annales de l'Institut Fourier*, 50(4):1021–1071, 2000.